Wandering triangles from the point of view of perturbations of postcritically finite maps

Jordi Canela
Institut de Mathématiques de Toulouse
Université Paul Sabatier

Joint work with: **Xavier Buff** and **Pascale Roesch**

Barcelona, 6 October 2017

Branch and wandering points

Definition

Let K be a connected and locally connected set. Then, w is a *branch point* if $K \setminus \{w\}$ has more than two components.

Definition

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map. We say that a point $w \in \mathcal{J}(f)$ is *wandering* if it has an infinite orbit.

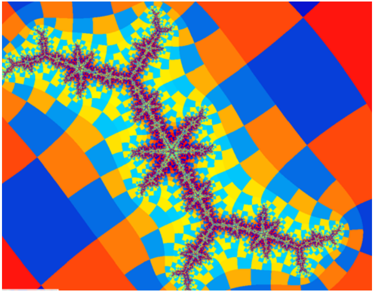
Branch and wandering points

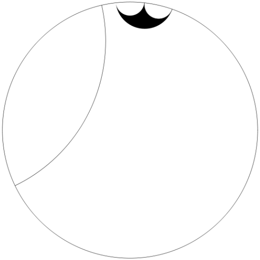
Definition
Let K be a connected and locally connected set. Then, w is a *branch point* if $K \setminus \{w\}$ has more than two components.

Definition
Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map. We say that a point $w \in \mathcal{J}(f)$ is *wandering* if it has an infinite orbit.

No wandering triangle

Thurston (1985): A branch point of a locally connected Julia set of a quadratic polynomial P is either eventually periodic or eventually critical.



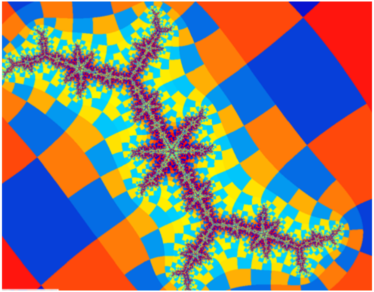


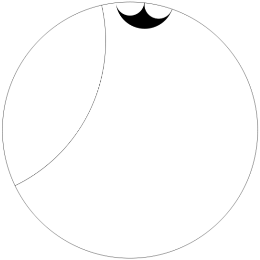
$\mathcal{J}(P)$ locally connected $\implies$ there is a lamination s.t. $\mathcal{J}(P) \simeq \overline{\mathbb{D}}/\sim$.

Thurston (1985): There is no wandering triangle in quadratic lamination.

No wandering triangle

Thurston (1985): A branch point of a locally connected Julia set of a quadratic polynomial P is either eventually periodic or eventually critical.



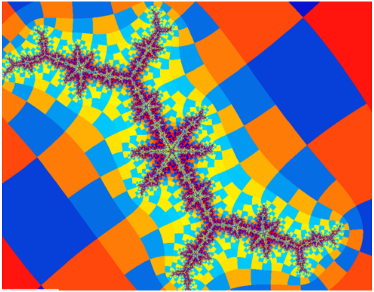


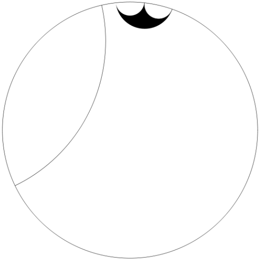
$\mathcal{J}(P)$ locally connected $\implies$ there is a lamination s.t. $\mathcal{J}(P) \simeq \overline{\mathbb{D}}/\sim$.

Thurston (1985): There is no wandering triangle in quadratic lamination.

No wandering triangle

Thurston (1985): A branch point of a locally connected Julia set of a quadratic polynomial P is either eventually periodic or eventually critical.



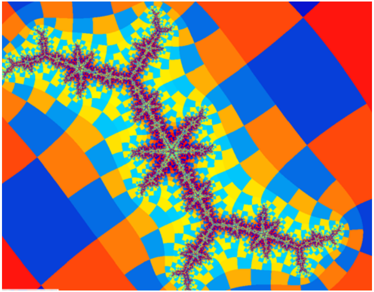


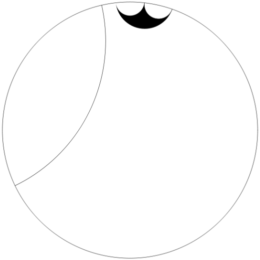
$\mathcal{J}(P)$ locally connected $\implies$ there is a lamination s.t. $\mathcal{J}(P) \simeq \overline{\mathbb{D}}/\sim$.

Thurston (1985): There is no wandering triangle in quadratic lamination.

No wandering triangle

Thurston (1985): A branch point of a locally connected Julia set of a quadratic polynomial P is either eventually periodic or eventually critical.

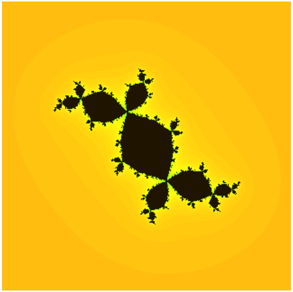




$\mathcal{J}(P)$ locally connected $\implies$ there is a lamination s.t. $\mathcal{J}(P) \simeq \overline{\mathbb{D}}/\sim$.

Thurston (1985): There is no wandering triangle in quadratic lamination.

In the case of locally connected Julia sets of polynomials,



w is a branch point $\iff$ $n \geq 3$ external rays land at w.

Wandering triangles: some results

- Thurston (1985):
  There is no wandering triangle in quadratic lamination.
  A branch point of a locally connected Julia set of a quadratic polynomial is either eventually periodic or eventually critical.
- Kiwi (2002): Every non-preperiodic non-precritical gap in a σ_d-invariant lamination is at most a d-gon.
  A wandering non pre-critical branch point of a degree d polynomial is the landing point of at most d external rays.
- Blokh (2005): If a cubic polynomial has wandering non-precritical points then the two critical points are recurrent one to each other.
- Blokh and Oversteegen (2008): There exist cubic polynomials with wandering non-precritical branch points.

Wandering triangles: some results

- Thurston (1985):
  There is no wandering triangle in quadratic lamination.
  A branch point of a locally connected Julia set of a quadratic polynomial is either eventually periodic or eventually critical.
- Kiwi (2002): Every non-preperiodic non-precritical gap in a σ_d-invariant lamination is at most a d-gon.
  A wandering non pre-critical branch point of a degree d polynomial is the landing point of at most d external rays.
- Blokh (2005): If a cubic polynomial has wandering non-precritical points then the two critical points are recurrent one to each other.
- Blokh and Oversteegen (2008): There exist cubic polynomials with wandering non-precritical branch points.

Wandering triangles: some results

- Thurston (1985):
  There is no wandering triangle in quadratic lamination.
  A branch point of a locally connected Julia set of a quadratic polynomial is either eventually periodic or eventually critical.
- Kiwi (2002): Every non-preperiodic non-precritical gap in a σ_d-invariant lamination is at most a d-gon.
  A wandering non pre-critical branch point of a degree d polynomial is the landing point of at most d external rays.
- Blokh (2005): If a cubic polynomial has wandering non-precritical points then the two critical points are recurrent one to each other.
- Blokh and Oversteegen (2008): There exist cubic polynomials with wandering non-precritical branch points.

Wandering triangles: some results

- Thurston (1985):
  There is no wandering triangle in quadratic lamination.
  A branch point of a locally connected Julia set of a quadratic polynomial is either eventually periodic or eventually critical.
- Kiwi (2002): Every non-preperiodic non-precritical gap in a σ_d-invariant lamination is at most a d-gon.
  A wandering non pre-critical branch point of a degree d polynomial is the landing point of at most d external rays.
- Blokh (2005): If a cubic polynomial has wandering non-precritical points then the two critical points are recurrent one to each other.
- Blokh and Oversteegen (2008): There exist cubic polynomials with wandering non-precritical branch points.

A concrete example

Buff-C.-Roesch: There exist a sequence of postcritically finite cubic polynomials (P_s) converging to a cubic polynomial with wandering non-precritical branch points.

- We start with a post-critical finite cubic polynomial of a certain type :

  one critical point is iterated to the other and finally to a periodic point uniquely accessible.

- we construct a sequence of polynomials of this type with critical points close to the initial ones but
  - with an increasing number of iterations
  - with the dynamical role of the critical points exchanged (there will be recurrent to each other)
- for each polynomial, some pre-image y_s of the critical point is separating 3 pre-periodic points
- At the limit the sequence (y_s) converges to a wandering non pre-critical branch point.

A concrete example

Buff-C.-Roesch: There exist a sequence of postcritically finite cubic polynomials (P_s) converging to a cubic polynomial with wandering non-precritical branch points.

- We start with a post-critical finite cubic polynomial of a certain type :

  one critical point is iterated to the other and finally to a periodic point uniquely accessible.

- we construct a sequence of polynomials of this type with critical points close to the initial ones but
  - with an increasing number of iterations
  - with the dynamical role of the critical points exchanged (there will be recurrent to each other)
- for each polynomial, some pre-image y_s of the critical point is separating 3 pre-periodic points
- At the limit the sequence (y_s) converges to a wandering non pre-critical branch point.

A concrete example

Buff-C.-Roesch: There exist a sequence of postcritically finite cubic polynomials (P_s) converging to a cubic polynomial with wandering non-precritical branch points.

- We start with a post-critical finite cubic polynomial of a certain type :

  one critical point is iterated to the other and finally to a periodic point uniquely accessible.

- we construct a sequence of polynomials of this type with critical points close to the initial ones but
  - with an increasing number of iterations
  - with the dynamical role of the critical points exchanged (there will be recurrent to each other)
- for each polynomial, some pre-image y_s of the critical point is separating 3 pre-periodic points
- At the limit the sequence (y_s) converges to a wandering non pre-critical branch point.

A concrete example

Buff-C.-Roesch: There exist a sequence of postcritically finite cubic polynomials (P_s) converging to a cubic polynomial with wandering non-precritical branch points.

- We start with a post-critical finite cubic polynomial of a certain type :

  one critical point is iterated to the other and finally to a periodic point uniquely accessible.

- we construct a sequence of polynomials of this type with critical points close to the initial ones but
  - with an increasing number of iterations
  - with the dynamical role of the critical points exchanged (there will be recurrent to each other)
- for each polynomial, some pre-image y_s of the critical point is separating 3 pre-periodic points
- At the limit the sequence (y_s) converges to a wandering non pre-critical branch point.

A concrete example

Buff-C.-Roesch: There exist a sequence of postcritically finite cubic polynomials (P_s) converging to a cubic polynomial with wandering non-precritical branch points.

- We start with a post-critical finite cubic polynomial of a certain type :

  one critical point is iterated to the other and finally to a periodic point uniquely accessible.

- we construct a sequence of polynomials of this type with critical points close to the initial ones but
  - with an increasing number of iterations
  - with the dynamical role of the critical points exchanged (there will be recurrent to each other)
- for each polynomial, some pre-image y_s of the critical point is separating 3 pre-periodic points
- At the limit the sequence (y_s) converges to a wandering non pre-critical branch point.

A concrete example

Buff-C.-Roesch: There exist a sequence of postcritically finite cubic polynomials (P_s) converging to a cubic polynomial with wandering non-precritical branch points.

- We start with a post-critical finite cubic polynomial of a certain type :

  one critical point is iterated to the other and finally to a periodic point uniquely accessible.

- we construct a sequence of polynomials of this type with critical points close to the initial ones but
  - with an increasing number of iterations
  - with the dynamical role of the critical points exchanged (there will be recurrent to each other)
- for each polynomial, some pre-image y_s of the critical point is separating 3 pre-periodic points
- At the limit the sequence (y_s) converges to a wandering non pre-critical branch point.

A concrete example

Buff-C.-Roesch: There exist a sequence of postcritically finite cubic polynomials (P_s) converging to a cubic polynomial with wandering non-precritical branch points.

- We start with a post-critical finite cubic polynomial of a certain type :

  one critical point is iterated to the other and finally to a periodic point uniquely accessible.

- we construct a sequence of polynomials of this type with critical points close to the initial ones but
  - with an increasing number of iterations
  - with the dynamical role of the critical points exchanged (there will be recurrent to each other)
- for each polynomial, some pre-image y_s of the critical point is separating 3 pre-periodic points
- At the limit the sequence (y_s) converges to a wandering non pre-critical branch point.

A concrete example

Buff-C.-Roesch: There exist a sequence of postcritically finite cubic polynomials (P_s) converging to a cubic polynomial with wandering non-precritical branch points.

- We start with a post-critical finite cubic polynomial of a certain type :

  one critical point is iterated to the other and finally to a periodic point uniquely accessible.

- we construct a sequence of polynomials of this type with critical points close to the initial ones but
  - with an increasing number of iterations
  - with the dynamical role of the critical points exchanged (there will be recurrent to each other)
- for each polynomial, some pre-image y_s of the critical point is separating 3 pre-periodic points
- At the limit the sequence (y_s) converges to a wandering non pre-critical branch point.

Set up

We consider monic cubic polynomials P with two critical points c_1 and c_2 such that $P(0) = 0$.

We say that P has an (n, m)-*configuration* if

- The external ray of angle 0 is the only ray landing at 0.
- There are $n, m > 0$ such that $P^n(c_2) = c_1$ and $P^m(c_1) = 0$.

$$c_2 \xrightarrow{P^n} c_1 \xrightarrow{P^m} 0$$

Note that the polynomials can be presented under the form

$$P_{c_1,c_2}(z) = z^3 - 3(c_1 + c_2)z^2/2 + 3c_1c_2z.$$

Set up

We consider monic cubic polynomials P with two critical points c_1 and c_2 such that $P(0) = 0$.

We say that P has an (n, m)-*configuration* if

- The external ray of angle 0 is the only ray landing at 0.
- There are $n, m > 0$ such that $P^n(c_2) = c_1$ and $P^m(c_1) = 0$.

$$c_2 \xrightarrow{P^n} c_1 \xrightarrow{P^m} 0$$

Note that the polynomials can be presented under the form

$$P_{c_1,c_2}(z) = z^3 - 3(c_1 + c_2)z^2/2 + 3c_1c_2z.$$

Set up

We consider monic cubic polynomials P with two critical points c_1 and c_2 such that $P(0) = 0$.

We say that P has an (n, m)-*configuration* if

- The external ray of angle 0 is the only ray landing at 0.
- There are $n, m > 0$ such that $P^n(c_2) = c_1$ and $P^m(c_1) = 0$.

$$c_2 \xrightarrow{P^n} c_1 \xrightarrow{P^m} 0$$

Note that the polynomials can be presented under the form

$$P_{c_1,c_2}(z) = z^3 - 3(c_1 + c_2)z^2/2 + 3c_1c_2z.$$

Set up

We consider monic cubic polynomials P with two critical points c_1 and c_2 such that $P(0) = 0$.

We say that P has an (n, m)-*configuration* if

- The external ray of angle 0 is the only ray landing at 0.
- There are $n, m > 0$ such that $P^n(c_2) = c_1$ and $P^m(c_1) = 0$.

$$c_2 \xrightarrow{P^n} c_1 \xrightarrow{P^m} 0$$

Note that the polynomials can be presented under the form

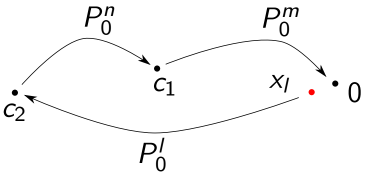
$$P_{c_1,c_2}(z) = z^3 - 3(c_1 + c_2)z^2/2 + 3c_1c_2z.$$

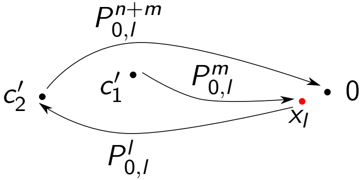
Perturbation of (n, m)-configurations

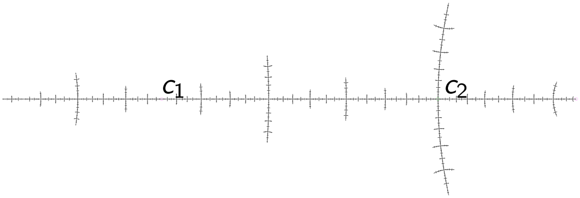
Lemma

Let P_0 be admissible with an (n, m)-configuration. Then, there are admissible polynomials $P_{0,l}$ such that:

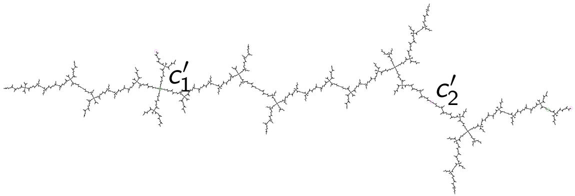
- *The polynomials $P_{0,l}$ converge to P_0 as $l \to \infty$.*
- *$P_{0,l}$ has critical points c_2' and c_1' which satisfy $P_{0,l}^{n+m}(c_2') = 0$ and $P_{0,l}^{m}(c_1') = x_l$. ($P_{0,l}$ has an $(m + l, m + n)$-configuration)*



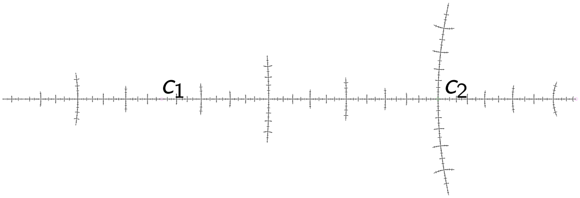




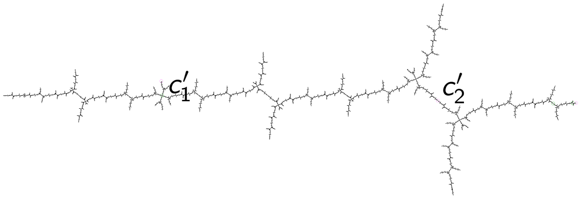
P_0: $n = m = 1$, $c_1 \approx -2.5980762113533$, $c_2 \approx -0.8660254037844$



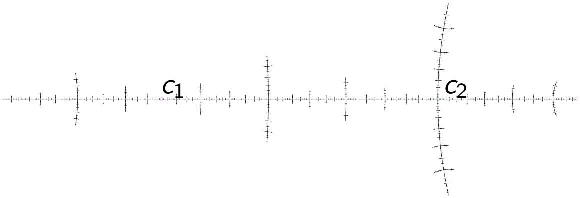
$P_{0,1}$: $n' = 2$, $m' = 2$, $c_1' \approx -2.5777842615361 + 0.1227176404951i$,
$c_2' \approx -0.8735669310080 + 0.0386710407537i$



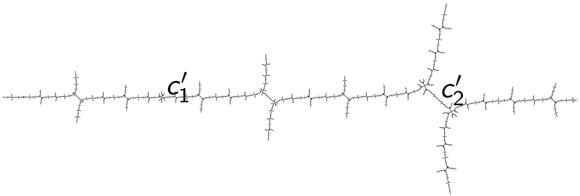
P_0: $n = m = 1$, $c_1 \approx -2.5980762113533$, $c_2 \approx -0.8660254037844$



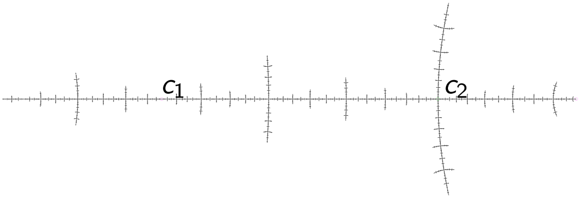
$P_{0,2}$: $n' = 3$, $m' = 2$, $c_1' \approx -2.5958326584619 + 0.0460203092748i$,
$c_2' \approx -0.8674089841015 + 0.0151384906087$



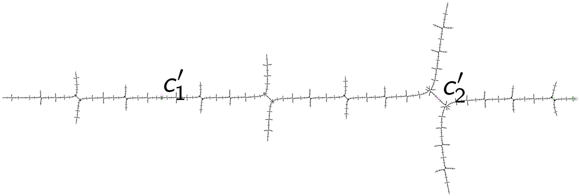
P_0: $n = m = 1$, $c_1 \approx -2.5980762113533$, $c_2 \approx -0.8660254037844$



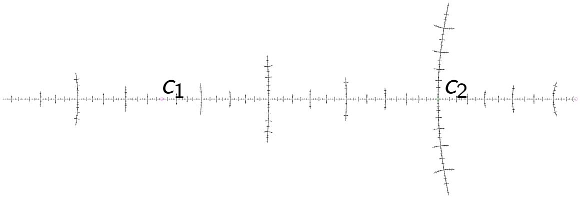
$P_{0,3}$: $n' = 4$, $m' = 2$, $c_1' \approx -2.5978601369971 + 0.0176164681833i$,
$c_2' \approx -0.8662705692505 + 0.0058563395448i$



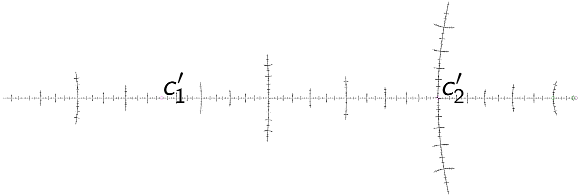
P_0: $n = m = 1$, $c_1 \approx -2.5980762113533$, $c_2 \approx -0.8660254037844$



$P_{0,4}$: $n' = 5$, $m' = 2$, $c_1' \approx -2.5980617426835 + 0.0067742534518i$, $c_2' \approx -0.8660676437257 + 0.0022569362336i$



P_0: $n = m = 1$, $c_1 \approx -2.5980762113533$, $c_2 \approx -0.8660254037844$

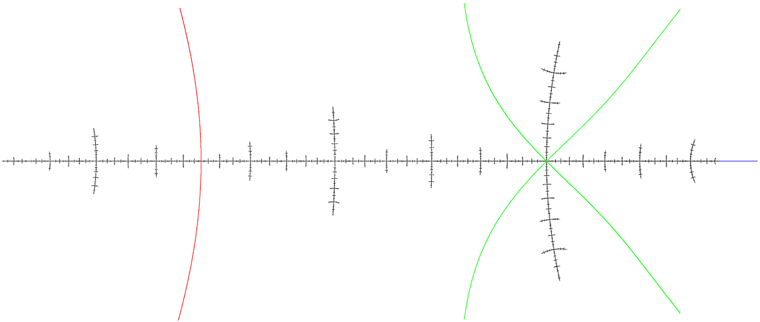


$P_{0,8}$: $n' = 9$, $m' = 2$, $c_1' \approx -2.5980762166193 + 0.0000572180661i$
$c_2' \approx -0.8660254089001 + 0.0000190726873i$,

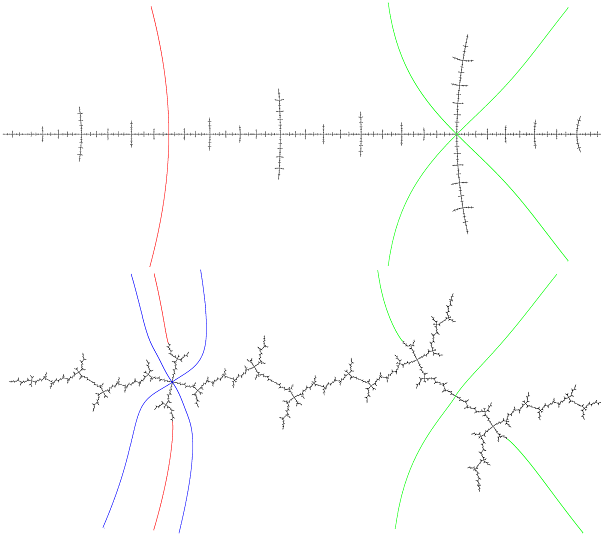
Pull back of the 0-ray at critical points

Assume that P has an (n, m)-configuration.

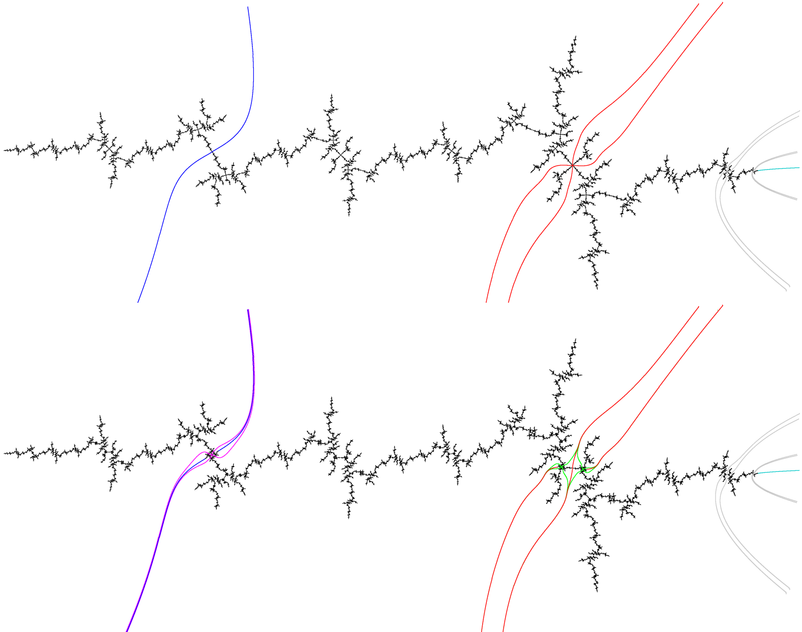
- 0 is the landing point of a single external ray.
- $P^m(c_1) = 0$. Hence, c_1 is the landing point of 2 pre-images.
- $P^n(c_2) = c_1$. Hence, c_2 is the landing point of 4 pre-images.



After perturbation



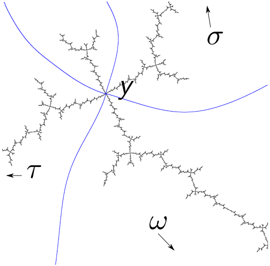
Rays near perturbed critical points

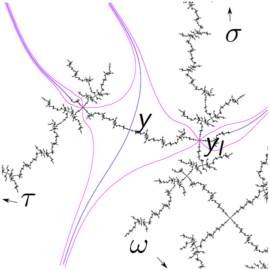


Separation of 3 points

Key lemma

Let P_0 be admissible with an $(n, m)-$ configuration. Let y be a preimage of c_2, $P_0^k(y) = c_2$, that separates preperiodic points ω, σ and τ.
Then, if $P_{0,l}$ is close enough to P_0, there exists a preimage y_l of c_1' which separates ω, σ and τ. Moreover, $P_{0,l}^{k+n}(y_l) = c_1'$.





Convergence of Carathéodory loops

The continuous extension $\overline{\psi}$ of the inverse ϕ^{-1} of the Böttcher map ϕ restricts to a continuous map $\gamma : \mathbb{S}^1 \to \mathcal{J}(P)$ called the Carathéodory loop.

Proposition

Let P_n be cubic polynomials with locally connected Julia set which converge to an admissible cubic polynomial P with an (n, m)-configuration. Then, the Carathéodory loops γ_n of P_n converge uniformly to the Carathéodory loop γ of P.

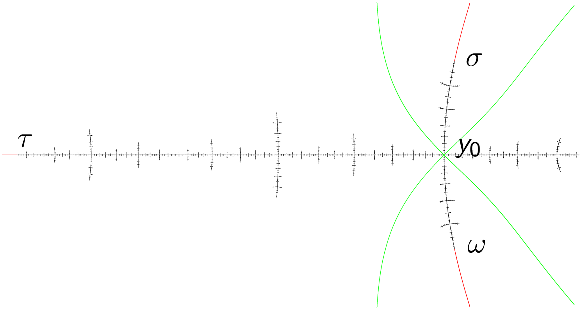
Convergence of Carathéodory loops

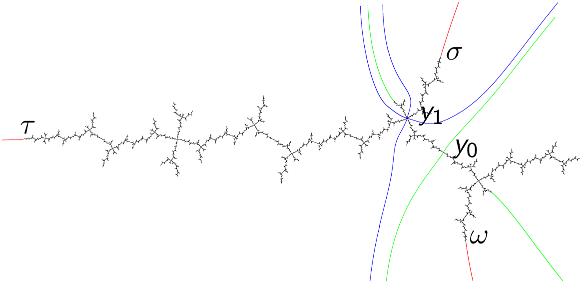
The continuous extension $\overline{\psi}$ of the inverse ϕ^{-1} of the Böttcher map ϕ restricts to a continuous map $\gamma : \mathbb{S}^1 \to \mathcal{J}(P)$ called the Carathéodory loop.

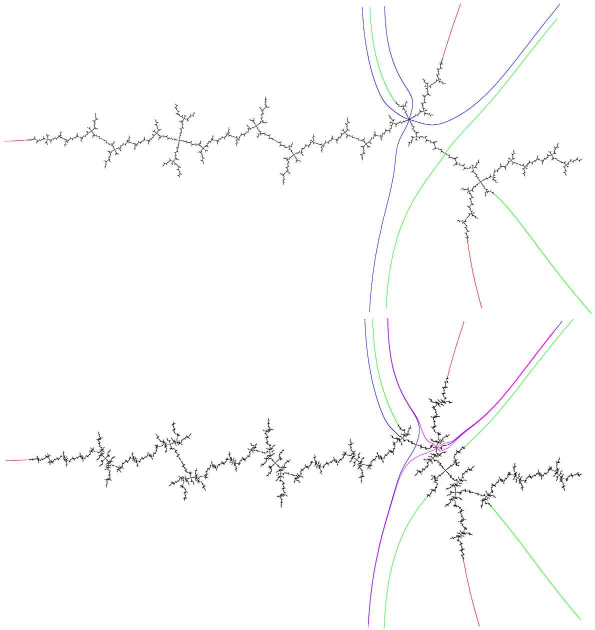
Proposition

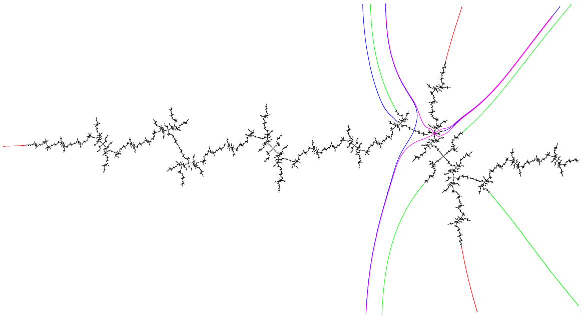
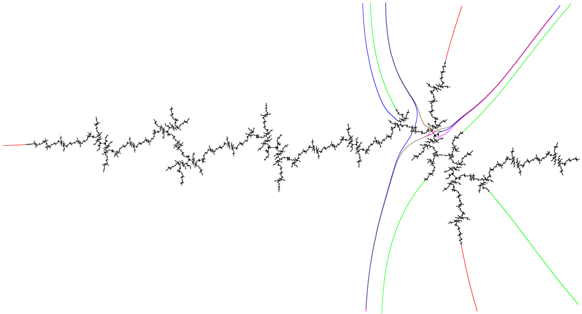
Let P_n be cubic polynomials with locally connected Julia set which converge to an admissible cubic polynomial P with an (n, m)-configuration. Then, the Carathéodory loops γ_n of P_n converge uniformly to the Carathéodory loop γ of P.

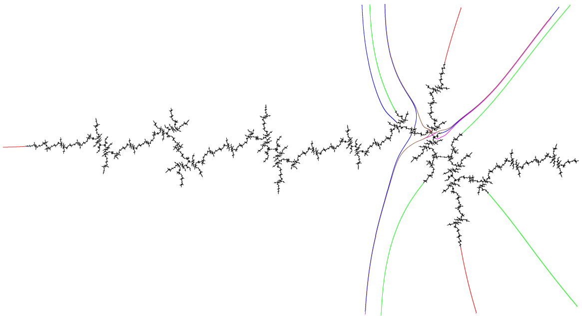
Iterative perturbations

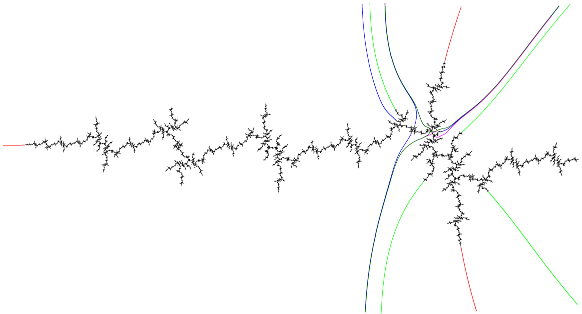


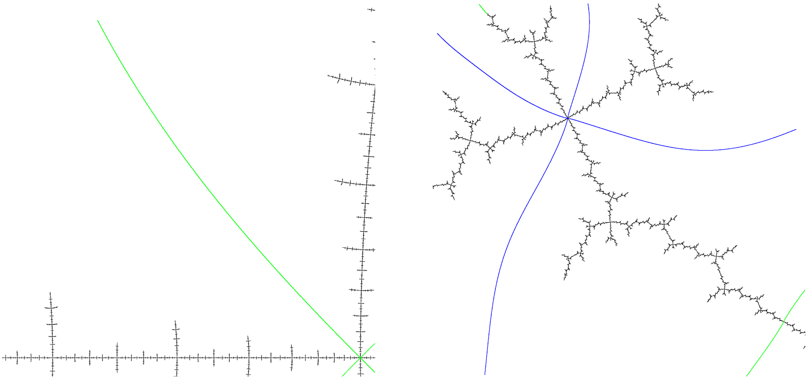


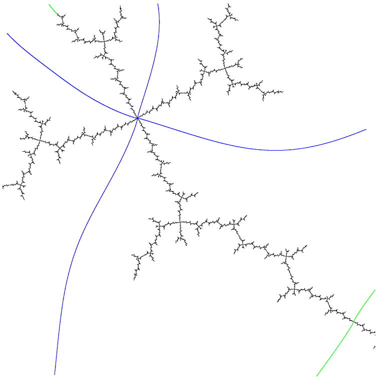
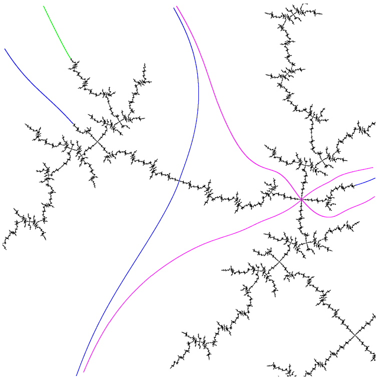


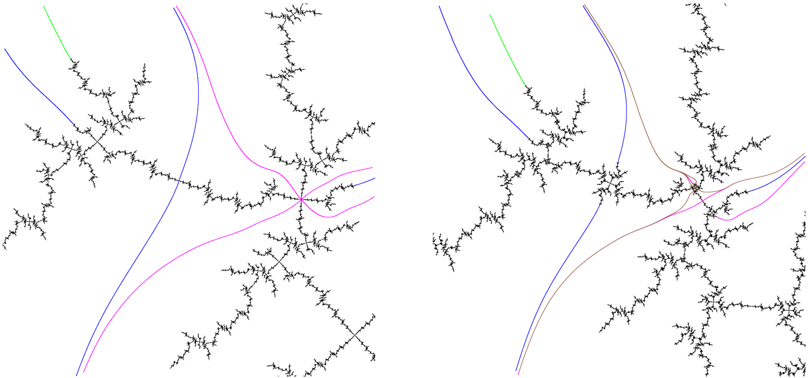



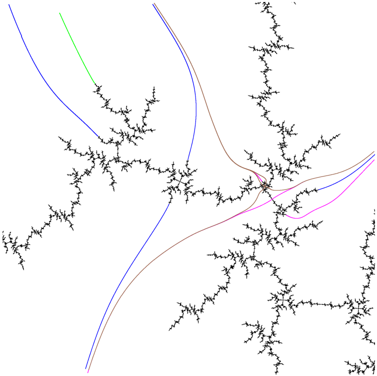


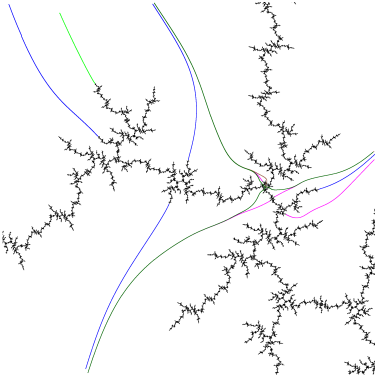










Thank you for your attention!