The Dynamics of Parabolic Transcendental Maps

Mashael Alhamd

University of Liverpool

October 3, 2017

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Another important set is the Escaping set, which is defined as follows

$$I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty \quad \text{as} \quad n \to \infty\}.$$

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- ∞ is a *superattracting* fixed point.
- By *Böttcher's theorem* there is a conformal map ϕ conjugating a polynomial f of degree $d \geq 2$ to $z \mapsto z^d$ near ∞.
- By Caratheodory -Torhorst Theorem the map ϕ^{-1} has a surjective continuous extension mapping $\partial\mathbb{D}$ to $\mathcal{J}(f)$ if and only if $\mathcal{J}(f)$ is connected.

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- Our goal is to extend these results to the setting of parabolic transcendental entire maps.

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- A complex number $\mathbf{v}$ called an attracting vector for f if $pa\mathbf{v}^p = 1$, and a repelling vector if $pa\mathbf{v}^p = -1$.
- Here the term "vector" should be thought of as a tangent vector to $\mathbb{C}$ at the origin. For example, as the tangent vector to the curve $t \mapsto t\mathbf{v}$ at $t = 0$.

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- $z \in \mathcal{P}$ if and only if there exists $N \in \mathbb{N}$ such that $f^k(z) \in \mathcal{P}$ for all $k \geq N$ via the vector $\mathbf{v}$ ($\mathrm{Arg}(f^k(z)) \to \mathrm{Arg}(v)$ for all $k \geq N$).

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Similarly, P is a repelling petal for f if it is an attracting petal for some local inverse g of f.

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[1]image source : https://commons.m.wikimedia.org

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This function has a parabolic fixed point at $z = 0$ with multiplier $f'(0) = 1$. It has four attracting (repelling) petals at zero.

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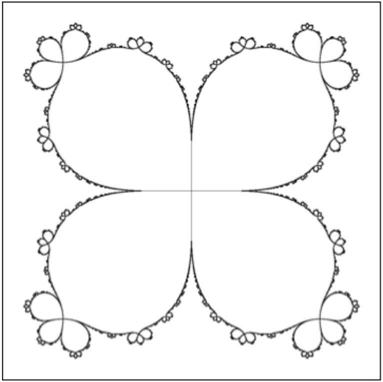

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  where $M(f,r)$ is the maximum absolute value of $f(z)$ where $|z| = r$.
- f has finite order if there exists $\rho, C > 0$ such that for all $r > 0$ $\sup_{|z|=r} |f(z)| \leq C.exp(r^\rho)$.

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Theorem[G. Rottenfusser, J. Ruckert, L. Rempe, and D. Schleicher]

If $f \in \mathcal{B}$ has finite order and of disjoint type. Then $\mathcal{J}(f)$ is a Cantor bouquet.

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Moreover, ϕ restricts to a homeomorphism between the escaping sets $I(g)$ and $I(f)$.

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The escaping set of a parabolic map is not connected.

hyperbolic vs parabolic Julia sets

[1] source of images: Lasse Rempe-Gillen

hyperbolic vs parabolic Julia sets

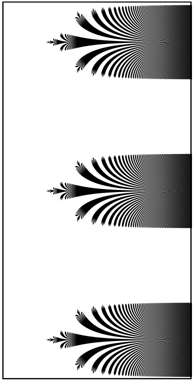

Figure: $f(z) = \frac{1}{2}(e^z - 1)$

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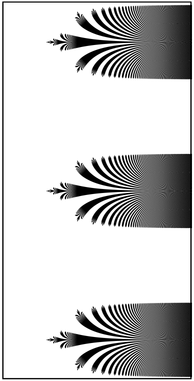

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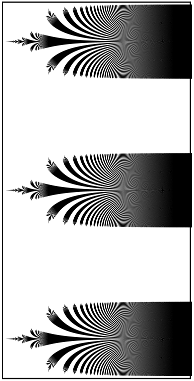

Figure: $g(z) = e^z - 1$

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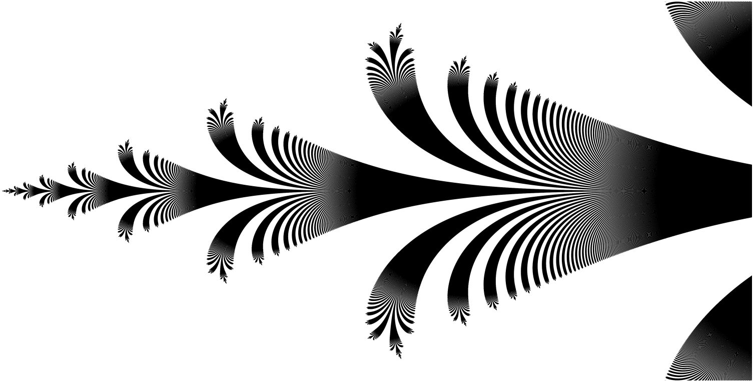

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Thank you !