Conjugacy classes of disjoint-type functions

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Example:

$$f(z) = z^2, \; g(z) = 2z^2 - 2z + 1, \; T(z) = 2z - 1$$

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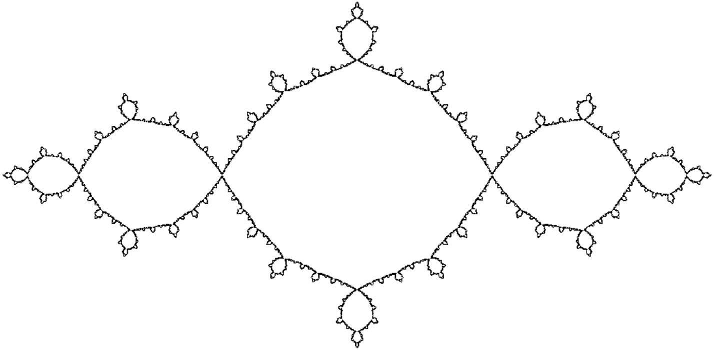

Source of image: Prokofiev, Wikimedia commons, `http://commons.wikimedia.org/wiki/File:Julia_z2-1.png`

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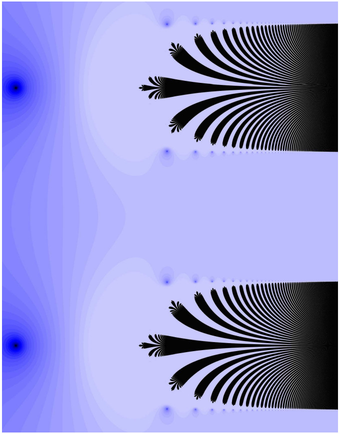

Image created by Lasse Rempe-Gillen.

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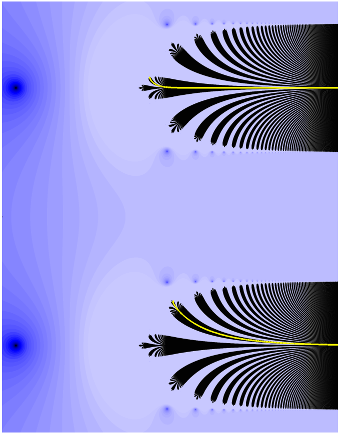

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Fact: All Cantor bouquets are homeomorphic to each other by ambient homeomorphisms (that is by homeomorphisms which can be extended to the whole plane).

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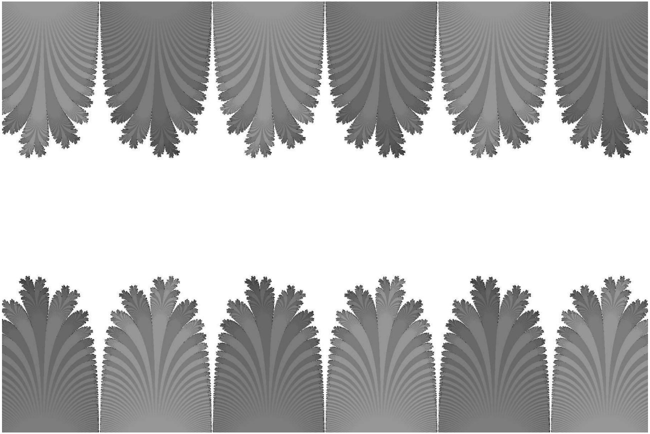

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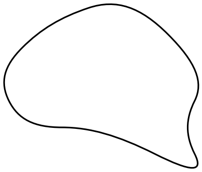

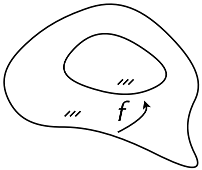
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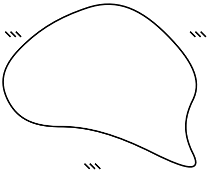

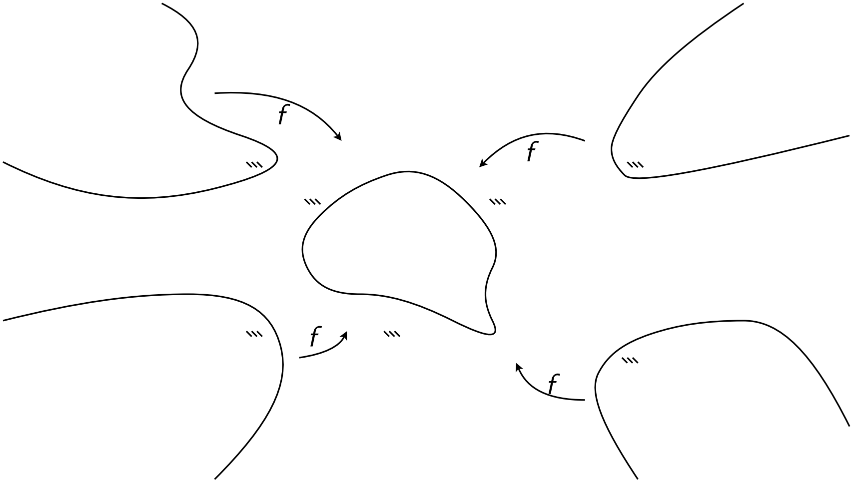
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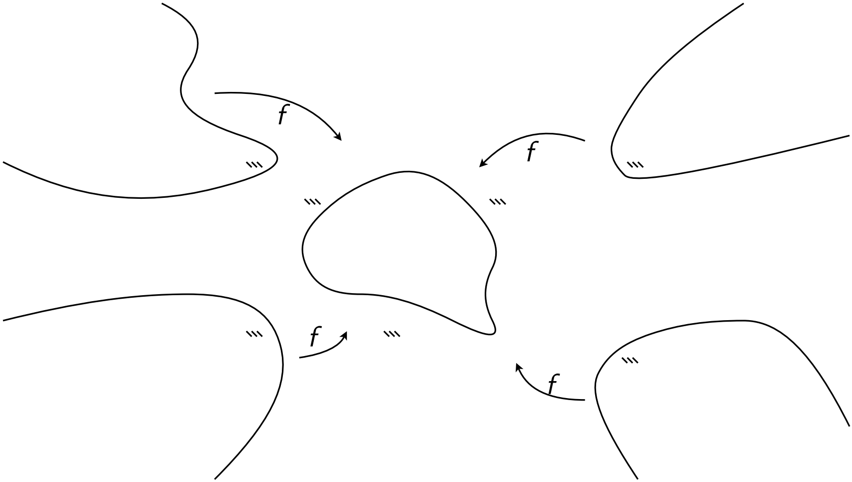

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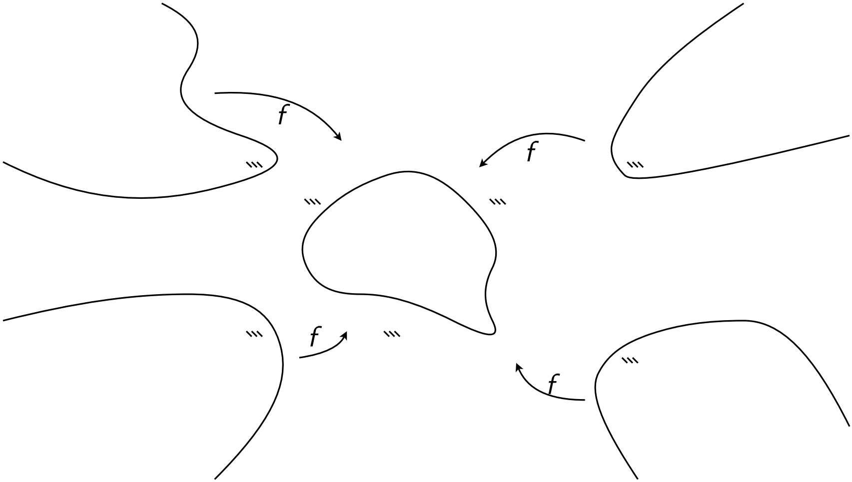

Example:

$e^z - 2$ is of disjoint type, e^z is not of disjoint type.

We say that an entire function f has *finite order*, if there exist $c, \rho > 0$ so that

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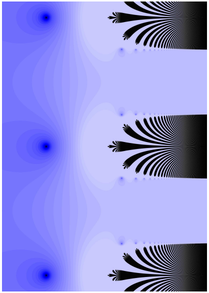

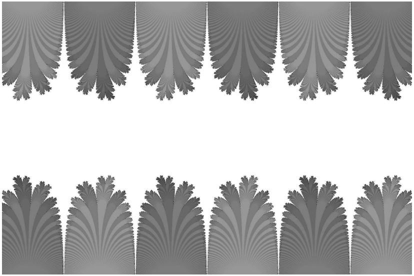

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However, there exist uncountably many disjoint-type functions of order 1 which are pairwise not conjugate on their Julia sets.

The conjugacy in general only holds on the Julia sets, not on the entire complex plane.

Thank you very much for your attention.