Rotation Sets
and
Complex Dynamics

Lecture III

November 26, 2015

1. Recap

• To each minimal rotation set X for $m_d : t \mapsto d \cdot t \pmod{\mathbb{Z}}$ we associate a ***rotation number*** $\rho(X) \in \mathbb{R}/\mathbb{Z}$ and ***deployment vector***

$$\delta(X) = (\delta_1, \ldots, \delta_{d-1}) \in \Delta^{d-2} \subset \mathbb{R}^{d-1},$$

where $\delta_i = \mu[z_{i-1}, z_i)$.

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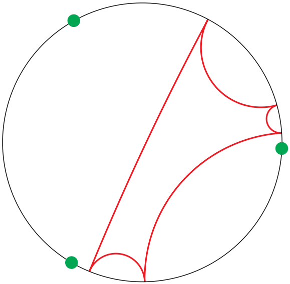
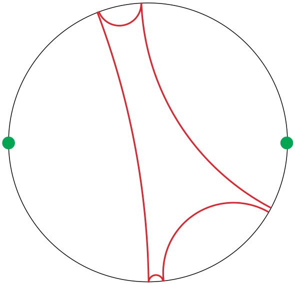


$\delta = \left(\frac{3}{5}, \frac{0}{5}, \frac{2}{5}\right)$



$\delta = (0.3819, 0.6181)$

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• Deployment Theorem: For any "admissible" pair $(\theta, \delta) \in (\mathbb{R}/\mathbb{Z}) \times \Delta^{d-2}$ there exists a unique minimal rotation set X for m_d such that $\rho(X) = \theta$ and $\delta(X) = \delta$.

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• The key tool is the ***gap measure***

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - k\theta},$$

which can be used to effectively construct X.

1. Recap

- Let ω denote the ***leading angle*** of X.

Theorem

$$\begin{aligned}\omega &= \frac{1}{d-1}\,\nu(0,\theta] + \frac{N_0}{d-1}\\ &= \frac{1}{d-1}\sum_{i=1}^{d-1}\sum_{0<\sigma_i-k\theta\leq\theta}\frac{1}{d^{k+1}} + \frac{N_0}{d-1}\end{aligned}$$

where $N_0 \geq 0$ is the number of 0's in $\sigma(X)$.

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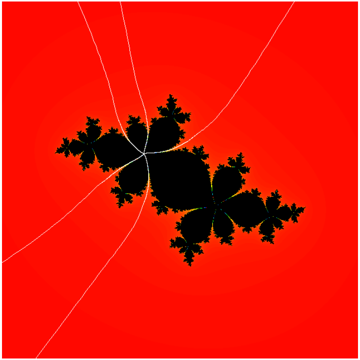
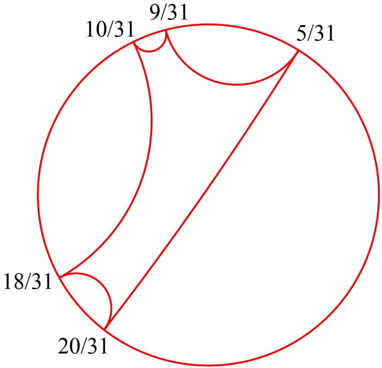
This gives an explicit algorithm for computing the base d expansion of the angle $(d-1)\omega$.

2. Rotation sets under doubling



$z \mapsto e^{2\pi i(2/5)}z + z^2$



$\rho(X) = 2/5$

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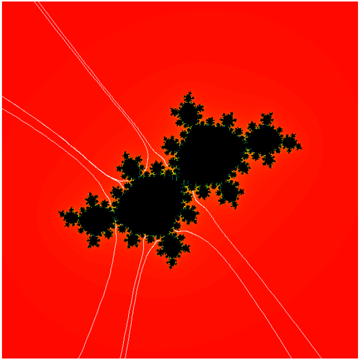
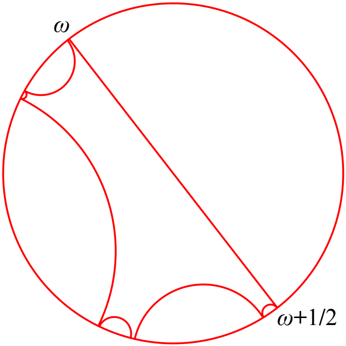


$$z \mapsto e^{2\pi i\theta}z + z^2$$



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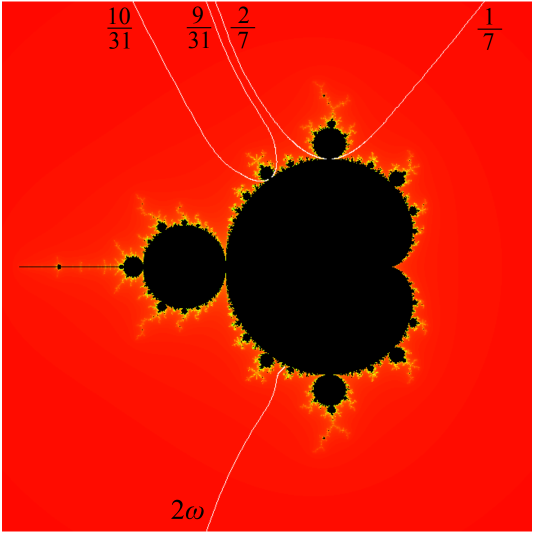

3. Rotation sets under tripling

Theorem

For every irrational number θ *and every* $\delta \in [0, 1]$ *there is a unique minimal rotation set* X *under the tripling map* $t \mapsto 3t \pmod{\mathbb{Z}}$ *with* $\rho(X) = \theta$ *and* $\delta(X) = (\delta, 1 - \delta)$.

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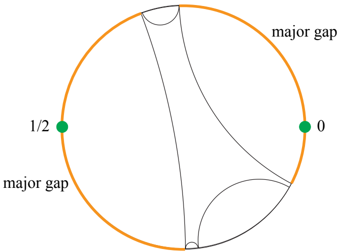

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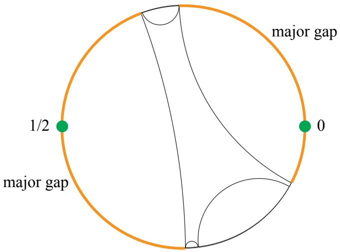

Notice that $X + 1/2$ is also a rotation set under tripling, with the same rotation number θ but deployment vector $(1-\delta, \delta)$.

3. Rotation sets under tripling

Theorem

There are countably many possibilities for the lengths of major gaps of X:

(i) *If* $\delta = 0 \pmod{\mathbb{Z}}$, *then* X *has a major gap of length* $\frac{2}{3}$.

(ii) *If* $\delta = \pm n\theta \pmod{\mathbb{Z}}$ *for some* $n \geq 1$, *then* X *has a pair of major gaps of lengths* $\frac{1}{3}$ *and* $\frac{1}{3} + \frac{1}{3^{n+1}}$.

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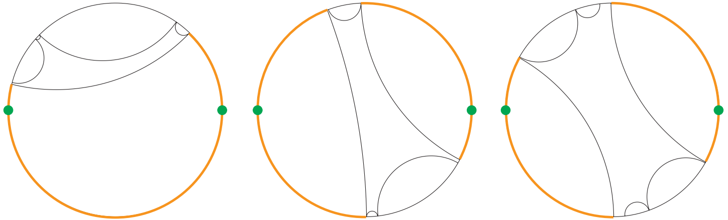

(i) (ii) (iii)

4. A cubic family

Fix an irrational number θ of bounded type and let $\lambda = e^{2\pi i\theta}$.

Definition

$\mathscr{P}(\lambda)$ is the space of conjugacy classes of cubic polynomials $\mathbb{C} \to \mathbb{C}$ with a fixed point of multiplier λ at the origin.

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Each element of $\mathscr{P}(\lambda)$ is represented by a monic polynomial of the form

$$f_a : z \mapsto \lambda z + az^2 + z^3 \qquad a \in \mathbb{C},$$

where a is uniquely determined up to the sign. Thus, the parameter

$$b = a^2 \in \mathbb{C}$$

is a complete invariant for the space $\mathscr{P}(\lambda)$.

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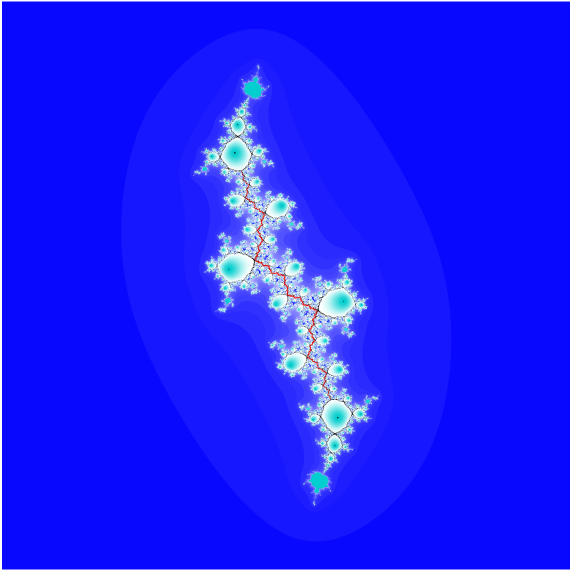

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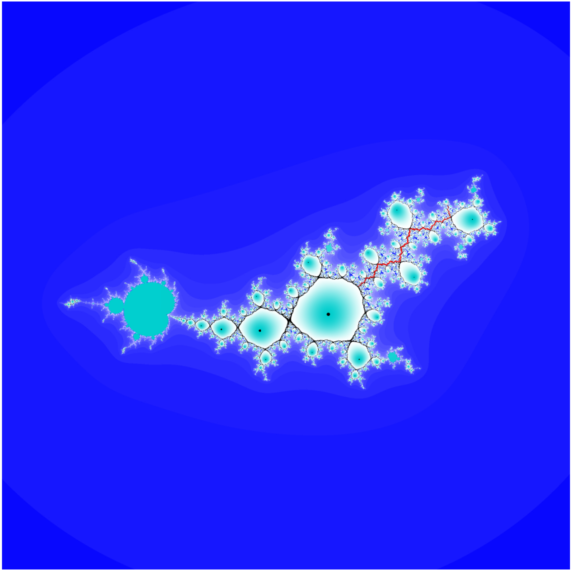

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5. Cubic Siegel disks

Since θ is bounded type, each f_a has a Siegel disk Δ_a centered at 0.

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For every $a \in \mathbb{C}$, the boundary $\partial\Delta_a$ is a quasicircle containing at least one critical point of f_a.

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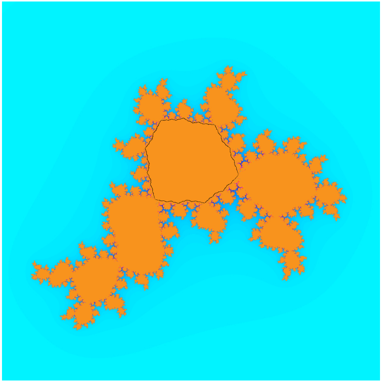

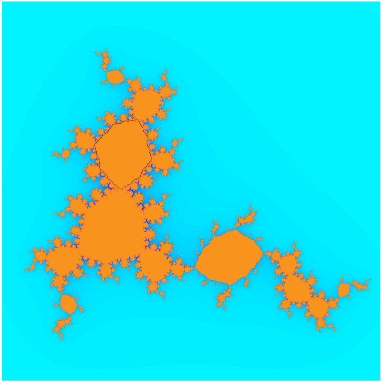

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Every interior component of $\mathscr{C}(\lambda)$ is of one of the following types:

- ***capture***, where the orbit of one critical point eventually hits the Siegel disk; or
- ***hyperbolic-like***, where the orbit of one critical point converges to an attracting cycle; or
- ***queer***, where the Julia set has positive measure and admits an invariant line field.

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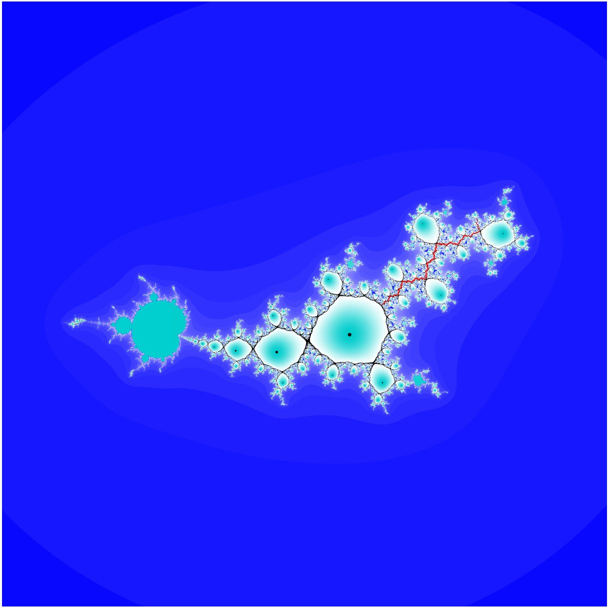

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Theorem

There is an embedded arc $\Gamma \subset \mathscr{C}(\lambda)$ connecting $b = 0$ to $b = 3\lambda$ with the property that $b = a^2 \in \Gamma$ if and only if $\partial \Delta_a$ contains both critical points of f_a.

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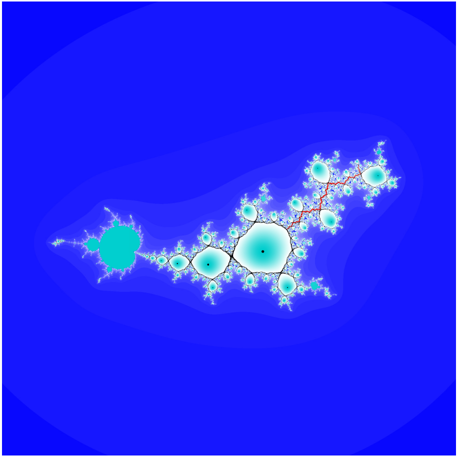

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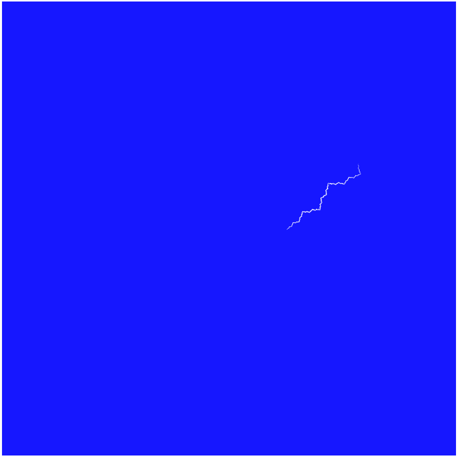

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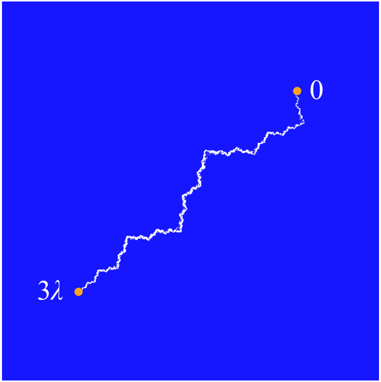

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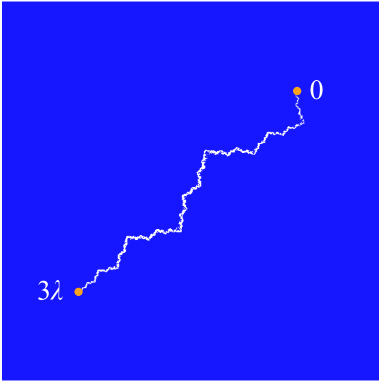

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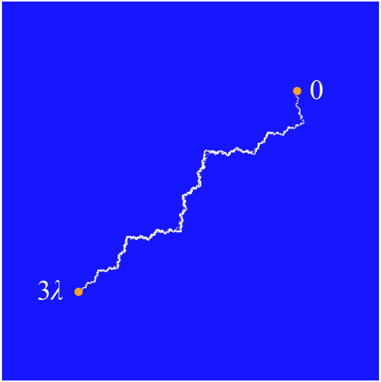

- Γ is parametrized by the conformal angle between the two critical points.
- Γ is the locus where the Siegel disk boundary fails to move holomorphically.

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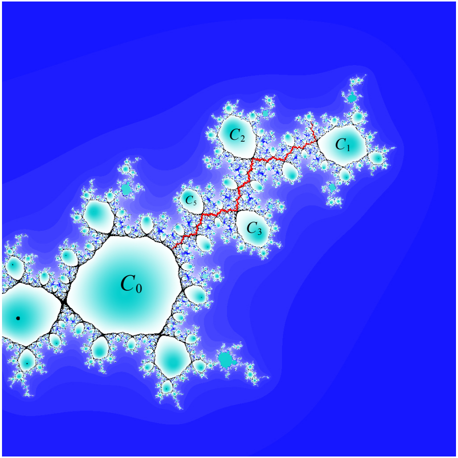

9. Dynamic and parameter rays

When $a \in \hat{\mathscr{C}}(\lambda)$, the Böttcher coordinate $\varphi_a(z) = z + O(1)$ of f_a defines the ***dynamic ray*** of f_a at angle t:

$$R_a(t) = \{\varphi_a^{-1}(re^{2\pi i t}) : r > 1\}.$$

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The map

$$\Phi : \mathbb{C} \setminus \hat{\mathscr{C}}(\lambda) \to \mathbb{C} \setminus \overline{\mathbb{D}} \qquad \Phi(a) = \varphi_a(c'_a)$$

is a conformal isomorphism. Since $\Phi(-a) = -\Phi(a)$, the map

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We define the ***parameter ray*** of $\mathscr{P}(\lambda)$ at angle t by

$$\mathcal{R}(t) = \{\Psi^{-1}(re^{2\pi i t}) : r > 1\}.$$

10. Limbs attached to Γ

There are two parameter rays $\mathcal{R}(t_{\mathbf{0}}^{-}), \mathcal{R}(t_{\mathbf{0}}^{+})$ landing at the root $b_{\mathbf{0}}$ of $C_{\mathbf{0}}$. They define the ***limb*** L_0 of $\mathscr{C}(\lambda)$ containing $C_{\mathbf{0}}$.

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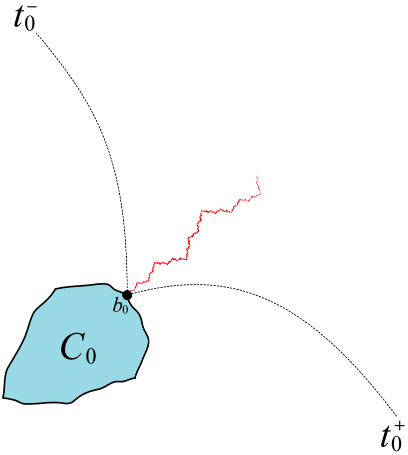

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For each $n \geq 1$, there are three parameter rays $\mathcal{R}(t_n^-), \mathcal{R}(t_n^+), \mathcal{R}(t_n^*)$ landing at the root b_n of C_n. The pair $\mathcal{R}(t_n^\pm)$ define the limb L_n of $\mathscr{C}(\lambda)$ containing C_n, while $\mathcal{R}(t_n^*)$ lies on the opposite side of Γ.

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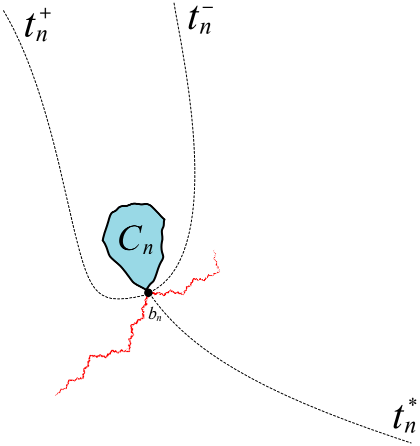

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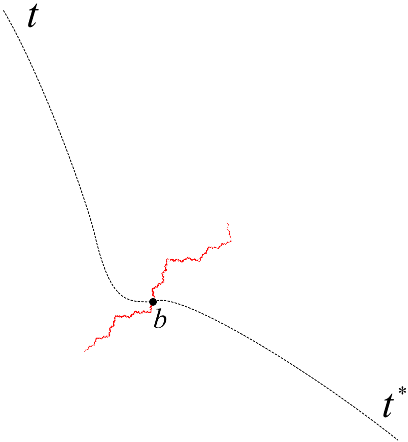

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The difference $X'_a \setminus X_a$ can have at most countably many isolated points, all eventually mapping to X_a under tripling.

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Let $(\delta_a, 1 - \delta_a)$ be the deployment vector of X_a.

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Notice that

$$X_{-a} = X_a + 1/2, \quad \text{so} \quad \delta_{-a} = 1 - \delta_a.$$

Thus, to each $b = a^2 \in \mathscr{C}(\lambda)$ we can assign a well-defined ***deployment probability*** $\delta_b \in [0, 1/2]$.

12. Deployment probability and Γ

Theorem

For every parameter $b = a^2 \in \Gamma$, δ_b is the conformal angle between the two critical points of f_a.

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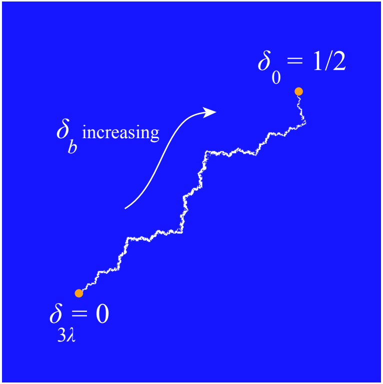

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- $\delta_b = 0 \pmod{\mathbb{Z}} \iff b \in L_{\mathbf{0}}$.
- $\delta_b = \pm n\theta \pmod{\mathbb{Z}}$ *for some* $n \geq 1 \iff b \in L_{\boldsymbol{n}}$.
- δ_b *takes other values* $\iff b \in \Gamma \setminus \{\text{roots points}\}$.

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The relation

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For the golden mean $\theta = (\sqrt{5}-1)/2$,

$$\omega = 0.128099593431\cdots$$

11. Computing angles

Theorem

We have

$$t_{\mathbf{0}}^{+} = 2\omega + \frac{2}{3},$$

and for $n \geq 1$*,*

$$\begin{cases} t_n^{+} & = \dfrac{(3^n + 1)\,\omega + p_n}{3^n} \\ t_n^{*} & = (3^n + 1)\,\omega - \dfrac{1}{3} \end{cases}$$

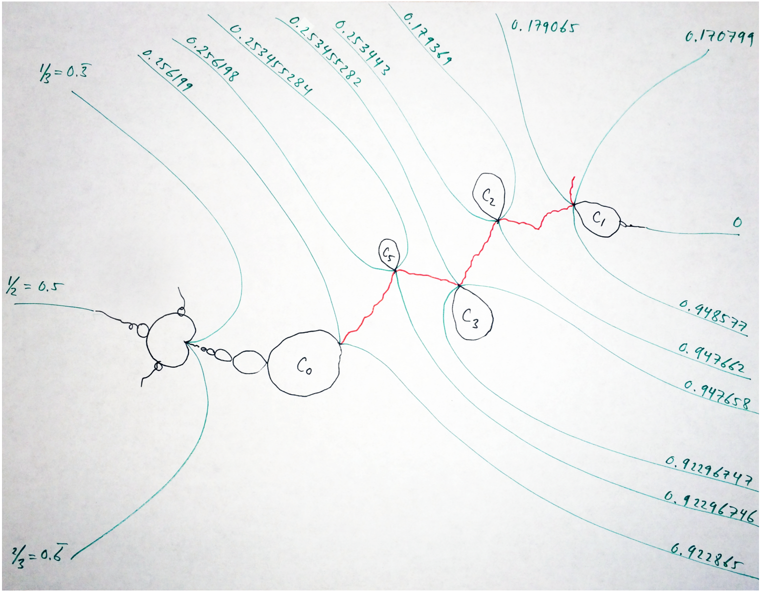
where $0 \leq p_n < 3^n$ *is a (computable) integer depending on* θ*.*

11. Computing angles

For example,

$$
\begin{aligned}
t_1^+ &= \frac{4\,\omega}{3} \\
t_2^+ &= \frac{10\,\omega + 1}{9} \\
t_3^+ &= \frac{28\,\omega + 22}{27} \\
t_4^+ &= \frac{82\,\omega + 4}{81} \\
t_5^+ &= \frac{244\,\omega + 31}{243}
\end{aligned}
$$

11. Computing angles



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