Rotation Sets and Complex Dynamics

Lecture II

November 24, 2015

A hyperbolic shower base

1. Recap

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• X has Lebesgue measure zero (hence is nowhere dense) on the circle.

• Each component of $\mathbb{T} \smallsetminus X$ is called a ***gap***. A gap of length ℓ is ***minor*** if $\ell < 1/d$ and ***major*** if $\ell \geq 1/d$. The multiplicity of a major gap is the integer part of $d \cdot \ell$. Counting multiplicities, X has precisely $d - 1$ major gaps.

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• There is a degree 1 monotone map $\varphi : \mathbb{T} \to \mathbb{T}$, normalized by $\varphi(0) = 0$, which satisfies

$$\varphi \circ m_d = r_\theta \circ \varphi \qquad \text{on } X$$

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and is constant on every gap of X. We simply refer to φ as the ***semiconjugacy associated with*** X.

• X carries a unique invariant probability measure μ which is related to φ by

$$\varphi(t) = \int_0^t d\mu = \mu[0, t].$$

μ is the uniform Dirac mass on X if $\rho(X)$ is rational, and the pull-back of Lebesgue measure under φ if $\rho(X)$ is irrational.

2. Deployment vector

Consider the $d - 1$ fixed points of m_d:

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If X is a rotation set for m_d with $\rho(X) \neq 0$, each major gap of multiplicity n contains exactly n fixed points of m_d.

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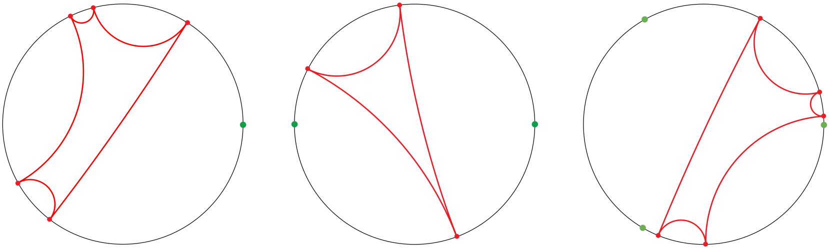

$d=2$ $d=3$ $d=4$

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Definition

The ***deployment vector*** of X is the probability vector

$$\delta(X) = (\delta_1, \ldots, \delta_{d-1})$$

where

$$\delta_i = \mu[z_{i-1}, z_i) \qquad 1 \leq i \leq d-1.$$

2. Deployment vector

The components of $\delta(X)$ can be interpreted as follows:

- If $\rho(X)$ is rational of the form p/q in lowest terms, then

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- If $\rho(X)$ is irrational, then

$$\delta_i = \lim_{n\to\infty} \frac{1}{n}\,\#\big\{0 \leq k \leq n-1 : m_d^{\circ k}(t) \in [z_{i-1}, z_i)\big\}$$

for every $t \in X$.

2. Deployment vector

We can also describe the deployment data by the ***cumulative deployment vector***

$$\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$$

where

$$\sigma_i = \delta_1 + \cdots + \delta_i = \mu[z_0, z_i).$$

Thus, $\sigma_1 \le \sigma_2 \le \cdots \le \sigma_{d-1} = 1$.

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Lemma

Suppose X is a minimal rotation set for m_d with $\sigma(X) = (\sigma_1, \ldots, \sigma_{d-1})$. Then

$$\sigma_i = \varphi(z_i) \pmod{\mathbb{Z}} \qquad 1 \leq i \leq d-1,$$

where φ is the semiconjugacy associated with X.

3. Deployment theorem: the rational case

Theorem (Goldberg)

For every rational number p/q *and every probability vector* $(\delta_1, \ldots, \delta_{d-1})$ *with* $q\delta_i \in \mathbb{Z}$*, there is a unique minimal rotation set* X *for* m_d *such that* $\rho(X) = p/q$ *and* $\delta(X) = (\delta_1, \ldots, \delta_{d-1})$.

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In particular, there is only one minimal rotation set under doubling of a given rational rotation number.

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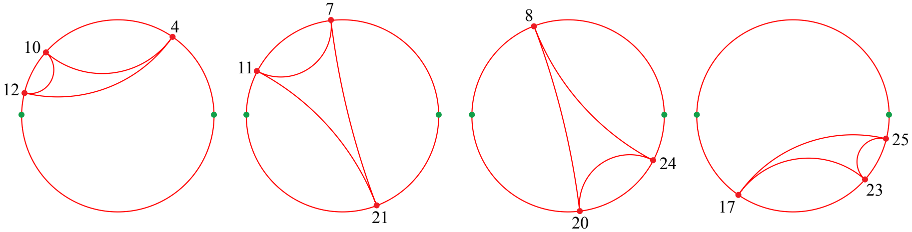
In particular, there is only one minimal rotation set under doubling of a given rational rotation number.

More generally, there are $\binom{q+d-2}{q}$ minimal rotation sets for m_d of a given rotation number p/q.

3. Deployment theorem: the rational case

- Example: $d = 3, \rho = 2/3$:



$\delta = (1,0)$ $\delta = \left(\frac{2}{3}, \frac{1}{3}\right)$ $\delta = \left(\frac{1}{3}, \frac{2}{3}\right)$ $\delta = (0,1)$

3. Deployment theorem: the rational case

• Example: Let us find the 5-cycle $X = \{t_1, \cdots, t_5\}$ under m_4 with $\rho(X) = \frac{1}{5}$ and $\delta(X) = (\frac{3}{5}, \frac{0}{5}, \frac{2}{5})$ or equivalently $\sigma(X) = (\frac{3}{5}, \frac{3}{5}, \frac{5}{5})$.

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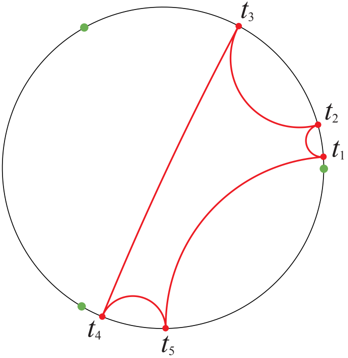

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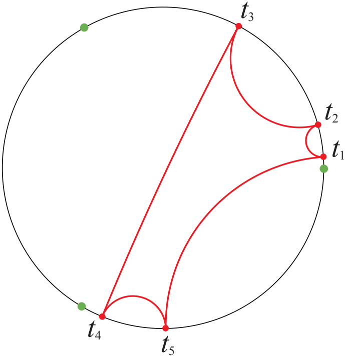

Let I_j be the gap (t_j, t_{j+1}).

I_3 is a major gap of multiplicity 2,

I_5 is a major gap of multiplicity 1,

and the remaining I_j's are minor.

Since $\rho(X) = \frac{1}{5}$, each I_j maps to I_{j+1}.

3. Deployment theorem: the rational case

If $\ell_j = |I_j|$, it follows that

$$\begin{aligned}
\ell_2 &= 4\ell_1 \\
\ell_3 &= 4\ell_2 = 4^2\ell_1 \\
\ell_4 &= 4\ell_3 - 2 = 4^3\ell_1 - 2 \\
\ell_5 &= 4\ell_4 = 4^4\ell_1 - 4\cdot 2 \\
\ell_1 &= 4\ell_5 - 1 = 4^5\ell_1 - 4^2\cdot 2 - 1.
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Hence,

$$\ell_1 = \frac{33}{1023}, \quad \ell_2 = \frac{132}{1023}, \quad \ell_3 = \frac{528}{1023}, \quad \ell_4 = \frac{66}{1023}, \quad \ell_5 = \frac{264}{1023}.$$

Since $\ell_1 = t_2 - t_1 = 4t_1 - t_1 = 3t_1$, we find t_1 and therefore every t_j:

$$t_1 = \frac{11}{1023}, \quad t_2 = \frac{44}{1023}, \quad t_3 = \frac{176}{1023}, \quad t_4 = \frac{704}{1023}, \quad t_5 = \frac{770}{1023}.$$

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More generally, suppose we want to find a minimal rotation set $X = \{t_1, \ldots, t_q\}$ under m_d with $\rho(X) = p/q \neq 0$ and $\delta(X) = (\delta_1, \ldots, \delta_{d-1})$.

Let ℓ_j denote the length of the gap $I_j = (t_j, t_{j+1})$ and $n_j \geq 0$ be the multiplicity of I_j. Then

$$\ell_{j+p} = d \cdot \ell_j - n_j \qquad \text{for all } j.$$

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Set

$$\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$$
$$\boldsymbol{n} = (n_1, \ldots, n_q)$$
$$T(x_1, x_2, \ldots, x_q) = (x_{1+p}, x_{2+p}, \ldots, x_{q+p}).$$

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Notice that T is determined by $\rho(X)$ while $\boldsymbol{n}$ is determined by $\delta(X)$.

3. Deployment theorem: the rational case

The q relations

$$\ell_{j+p} = d \cdot \ell_j - n_j \qquad 1 \le j \le q$$

can now be written as

$$T(\boldsymbol{\ell}) = d\,\boldsymbol{\ell} - \boldsymbol{n}.$$

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Using $T^{\circ q} = \mathrm{id}$, this can be easily solved for $\boldsymbol{\ell}$:

$$\boldsymbol{\ell} = \frac{1}{d^q - 1} \sum_{i=0}^{q-1} d^{q-i-1}\, T^{\circ i}(\boldsymbol{n}).$$

Since $\boldsymbol{n} \neq \mathbf{0}$, the solution $\boldsymbol{\ell}$ has positive components ℓ_j for all j.

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Once the gap lengths ℓ_j are known, we can find the t_j by noting that the counterclockwise distance from t_j to $t_{j+p} = d \cdot t_j \pmod{\mathbb{Z}}$ is the sum $\ell_j + \cdots + \ell_{j+p-1}$.

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Alternatively,

$$\ell = \sum_{k=0}^{\infty} d^{-(k+1)}\, T^{\circ k}(\boldsymbol{n}).$$

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The vectors $\boldsymbol{\ell}$ and $\boldsymbol{n}$ can be thought of as atomic measures supported on $\{1/q, \ldots, q/q\} \subset \mathbb{T}$ by identifying ℓ_j with $\boldsymbol{\ell}\{j/q\}$ and n_j with $\boldsymbol{n}\{j/q\}$.

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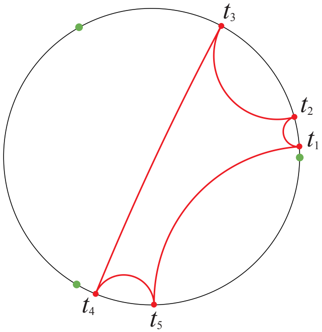

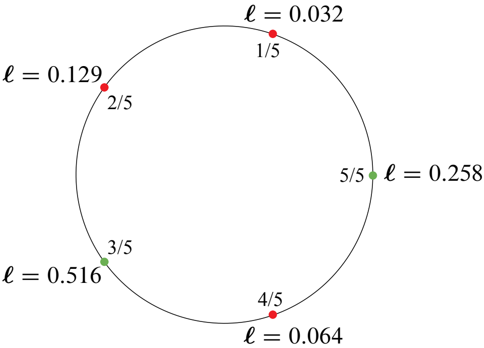

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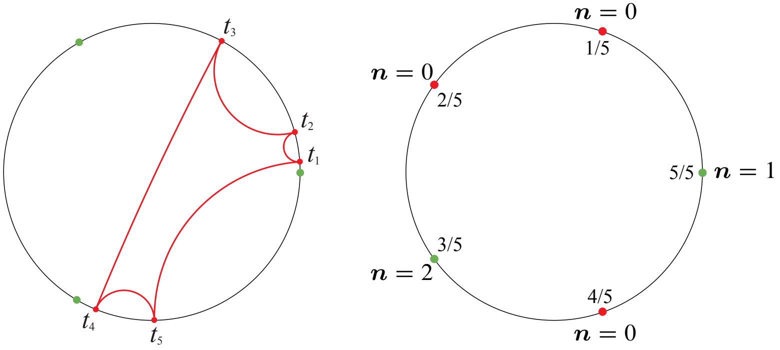

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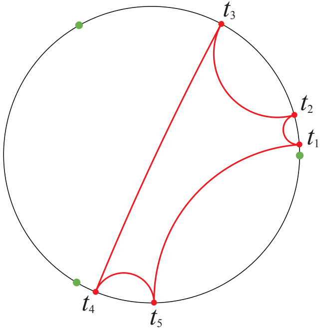
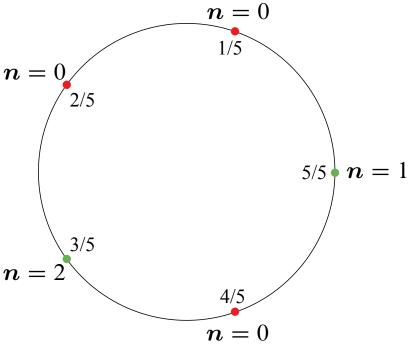
Under this identification, $\boldsymbol{\ell}$ is just the push-forward of Lebesgue measure under the semiconjugacy associated with X, and

$$\boldsymbol{n} = \sum_{i=1}^{d-1} \mathbb{1}_{\sigma_i}\,,$$





3. Deployment theorem: the rational case

Thus, for each $k \geq 0$,

$$T^{\circ k}(\boldsymbol{n}) = \sum_{i=1}^{d-1} \mathbb{1}_{\sigma_i - kp/q}$$

and the alternative formula

$$\ell = \sum_{k=0}^{\infty} d^{-(k+1)}\, T^{\circ k}(\boldsymbol{n}).$$

can be written as

$$\ell = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - kp/q}.$$

4. Deployment theorem: the irrational case

Theorem (Goldberg-Tresser)

For every irrational number θ and every probability vector $(\delta_1, \ldots, \delta_{d-1})$, there is a unique minimal rotation set X for m_d such that $\rho(X) = \theta$ and $\delta(X) = (\delta_1, \ldots, \delta_{d-1})$.

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In particular, there is only one minimal rotation set under doubling of a given irrational rotation number.

More generally, the space of all rotation sets for m_d of a given irrational rotation number is isomorphic to the $(d-2)$-dimensional simplex $\Delta^{d-2} \subset \mathbb{R}^{d-1}$.

4. Deployment theorem: the irrational case

Idea of the proof:

- Consider the ***gap measure***

$$\nu = \sum_{i=1}^{d-1} \sum_{k=0}^{\infty} d^{-(k+1)} \mathbb{1}_{\sigma_i - k\theta},$$

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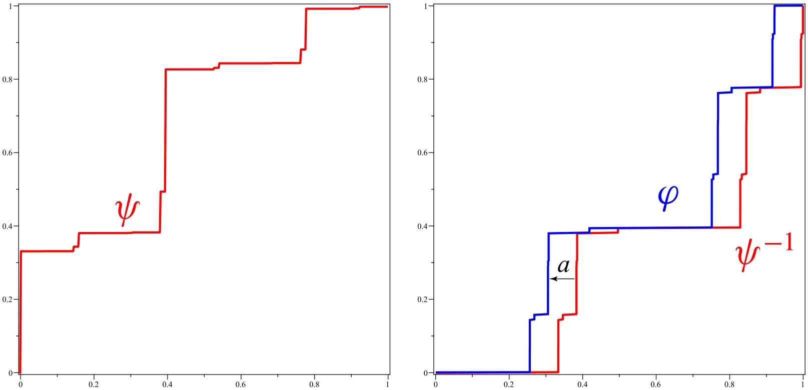
- The semiconjugacy associated with X will be

$$\varphi(t) = \psi^{-1}(t+a)$$

for suitable a.

4. Deployment theorem: the irrational case



$$\text{Here } d = 3 \text{ and } \begin{cases} \rho(X) & = (\sqrt{5}-1)/2 \\ \delta(X) & = (0.39475, 0.60525) \\ \quad a & = 0.07713 \end{cases}$$

5. Some corollaries

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Let $A \subset \mathbb{T} \times \Delta^{d-2}$ be the set of all pairs (θ, δ) subject to the restriction that if $\theta = p/q$ in lowest terms, then $q\delta \in \mathbb{Z}^{d-1}$. For each $(\theta, \delta) \in A$, let $X_{\theta,\delta}$ be the unique minimal rotation set for m_d with $\rho(X_{\theta,\delta}) = \theta$ and $\delta(X_{\theta,\delta}) = \delta$.

Theorem

The assignment $(\theta, \delta) \mapsto X_{\theta,\delta}$ from A to the space of compact subsets of the circle is continuous at (θ, δ) if and only if $X_{\theta,\delta}$ is rigid.

5. Some corollaries

Let ω denote the ***leading angle*** of $X_{\theta,\delta}$.

Theorem

$$\omega = \frac{1}{d-1}\,\nu(0,\theta] + \frac{N_0}{d-1}$$
$$= \frac{1}{d-1}\sum_{i=1}^{d-1}\sum_{0<\sigma_i-k\theta\leq\theta}\frac{1}{d^{k+1}} + \frac{N_0}{d-1}$$

where $N_0 \geq 0$ *is the number of* 0*'s in* $\sigma(X)$.