Rotation Sets and Complex Dynamics

Lecture I

November 23, 2015

1. Motivation

For every rational number p/q there is a unique periodic orbit in $\mathbb{R}/\mathbb{Z}$ under the doubling map $t \mapsto 2t \pmod{\mathbb{Z}}$ whose rotation number is p/q:

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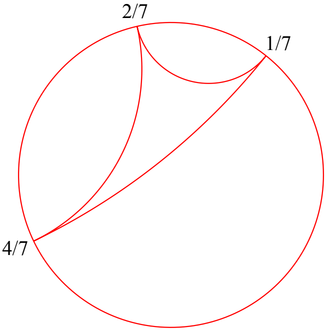
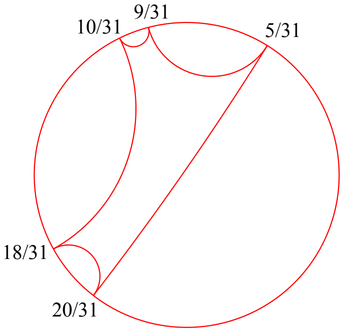


$p/q = 1/3$



$p/q = 2/5$

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Similarly, for every irrational number θ, there is a unique compact invariant (Cantor) set in $\mathbb{R}/\mathbb{Z}$ whose rotation number under doubling is θ:

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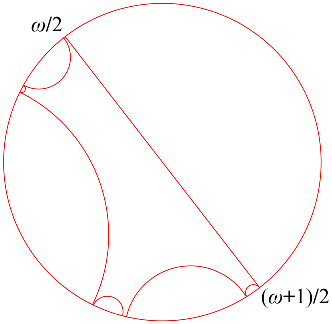
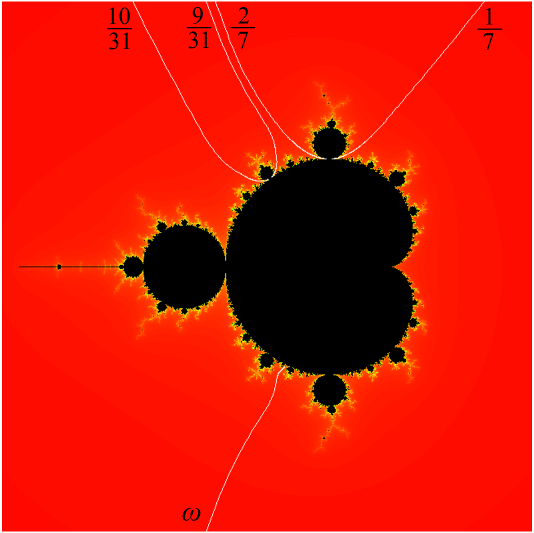


$$\theta = (\sqrt{5} - 1)/2 \qquad \omega = 0.7098034428\cdots$$

1. Motivation

These “rotation sets” under doubling describe angles of the external rays that land on the boundary of the main cardioid of the Mandelbrot set:



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- Abstract part: Classification of rotation sets under multiplication by $d \geq 2$.

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- Abstract part: Classification of rotation sets under multiplication by $d \geq 2$.
- Concrete part: Realizing rotation sets in suitable spaces of degree d polynomials.

2. Monotone maps of the circle

• $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the unit circle

• A map $g : \mathbb{T} \to \mathbb{T}$ is ***degree*** 1 ***monotone*** if it lifts to $G : \mathbb{R} \to \mathbb{R}$ which is non-decreasing and satisfies $G(x + 1) = G(x) + 1$ for all x.

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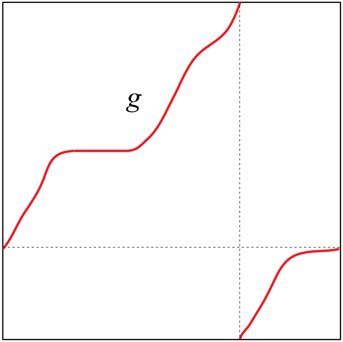

2. Monotone maps of the circle

• Consider the sets

$$A^- = \left\{ \frac{p}{q} : G^{\circ q}(x) > x + p \text{ for all } x \right\}$$
$$A^+ = \left\{ \frac{p}{q} : G^{\circ q}(x) < x + p \text{ for all } x \right\},$$

where p, q are integers with $q > 0$.

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where p, q are integers with $q > 0$.

- The pair (A^{-}, A^{+}) is a Dedekind cut of $\mathbb{Q}$:

2. Monotone maps of the circle

- The ***translation number*** of G is defined as

$$\tau(G) = \sup A^{-} = \inf A^{+}.$$

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$$\tau(G) = \lim_{n\to\infty} \frac{G^{\circ n}(x) - x}{n} \qquad \text{for any } x \in \mathbb{R}.$$

Thus, $\tau(G)$ measures the average translation per iterate that each point experiences under repeated applications of G.

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Thus, $\tau(G)$ measures the average translation per iterate that each point experiences under repeated applications of G.

Definition

The ***rotation number*** $\rho(g)$ is the residue class modulo $\mathbb{Z}$ of the translation number $\tau(G)$, often identified with its representative in $[0, 1)$.

2. Monotone maps of the circle

- Example: For $0 \leq \theta < 1$, the ***rigid rotation***

$$r_\theta(t) = t + \theta \pmod{\mathbb{Z}}$$

has rotation number $\rho(r_\theta) = \theta$.

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• Example: For $0 \le \theta < 1$, the ***rigid rotation***

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has rotation number $\rho(r_\theta) = \theta$.

• Rotation number determines cyclic order of orbit points: If $\rho(g) = \theta$, and if the triple

$$r_\theta^{\circ i}(0), \quad r_\theta^{\circ j}(0), \quad r_\theta^{\circ k}(0)$$

has positive cyclic order, so does

$$g^{\circ i}(t), \quad g^{\circ j}(t), \quad g^{\circ k}(t)$$

for every $t \in \mathbb{T}$.

2. Monotone maps of the circle

Theorem

Suppose $\rho(g) = p/q$ in lowest terms. Then,

- *(i) g has a periodic orbit of length q.*
- *(ii) All periodic orbits of g have length q.*
- *(iii) If the points of a periodic orbit are labeled in positive cyclic order as $t_1, \ldots, t_q$, then $g(t_j) = t_{j+p}$.*
- *(iv) $\omega(t)$ is a periodic orbit for every $t \in \mathbb{T}$.*

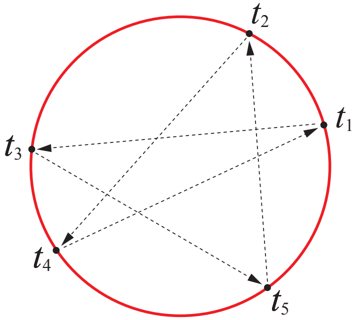
Recall that

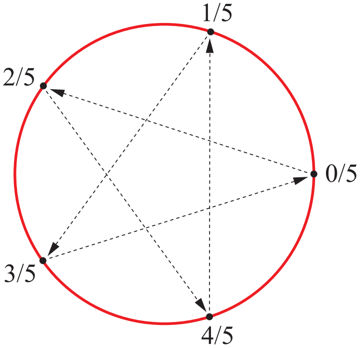
$$\omega(t) = \bigcap_{n \geq 1} \overline{\{g^{\circ n}(t), g^{\circ n+1}(t), g^{\circ n+2}(t), \ldots\}}$$

is the set of all accumulation points of the g-orbit of t.

2. Monotone maps of the circle

- Example: $\rho(g) = 2/5$





The 5-cycle $C = \{t_1, \ldots, t_5\}$ has ***combinatorial rotation number*** $2/5$.

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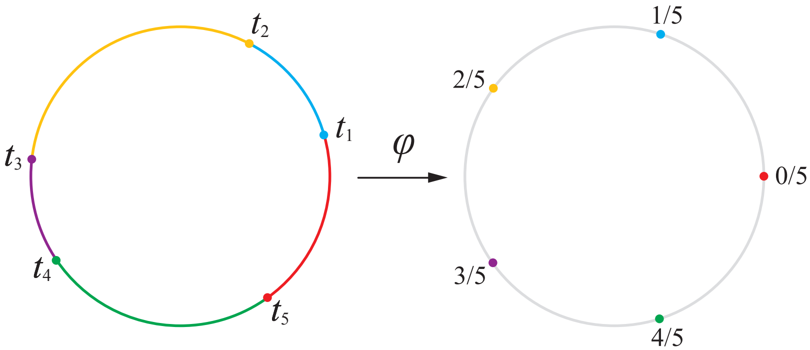

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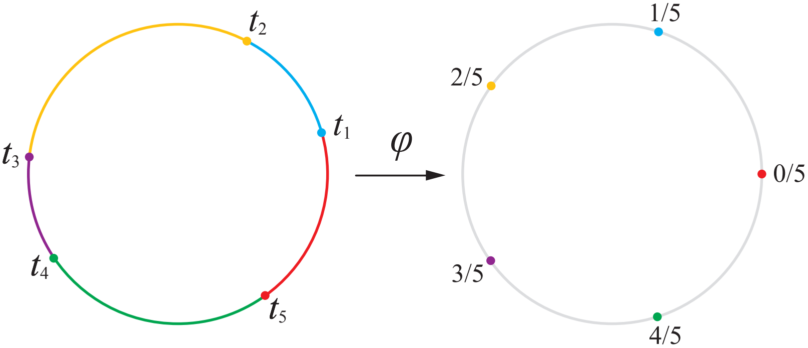

We call φ the ***combinatorial semiconjugacy*** associated with the cycle C:

$$\varphi \circ g = r_{p/q} \circ \varphi \qquad \text{on } C.$$

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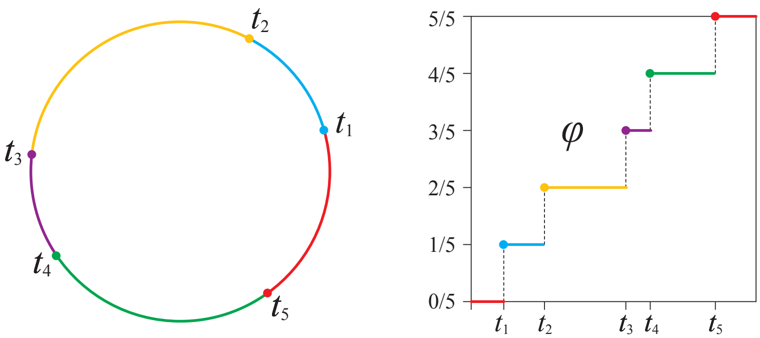

The cycle C is the complement of the union of the “plateaus” of φ.

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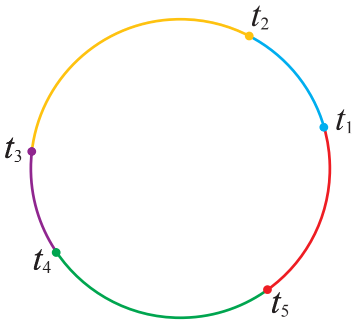

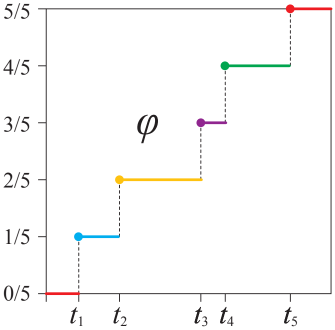

If μ is the unique invariant measure supported on C, then

$$\varphi(t) = \mu[0, t].$$

2. Monotone maps of the circle

Now suppose $\rho(g) = \theta$ is irrational.

Theorem (Poincaré)

There exists a degree 1 *monotone map* $\varphi : \mathbb{T} \to \mathbb{T}$ *such that* $\varphi \circ g = r_\theta \circ \varphi$*. Moreover,* φ *is unique up to postcomposition with a rigid rotation.*

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Now suppose $\rho(g) = \theta$ is irrational.

Theorem (Poincaré)

There exists a degree 1 monotone map $\varphi : \mathbb{T} \to \mathbb{T}$ such that $\varphi \circ g = r_\theta \circ \varphi$. Moreover, φ is unique up to postcomposition with a rigid rotation.

We call the map φ normalized by $\varphi(0) = 0$ the ***Poincaré semiconjugacy*** between g and r_θ.

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "g"] \arrow[d, "\varphi"'] & \mathbb{T} \arrow[d, "\varphi"] \\
\mathbb{T} \arrow[r, "r_\theta"] & \mathbb{T}
\end{tikzcd}$$

2. Monotone maps of the circle

Theorem

Suppose φ is the Poincaré semiconjugacy between g and r_θ:

(i) If φ is a homeomorphism, then $\omega(t) = \mathbb{T}$ for all $t \in \mathbb{T}$.

(ii) If φ is not a homeomorphism, there is a g-invariant Cantor set K such that $\omega(t) = K$ for every $t \in \mathbb{T}$.

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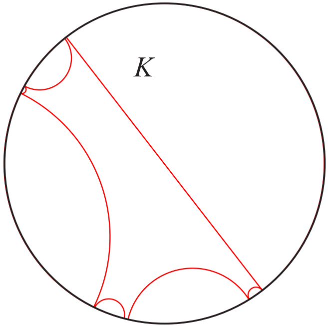
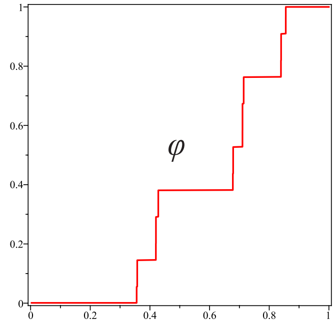
The compact set K in case (ii) is called the ***Cantor attractor*** of g. It can be described as the complement of the union of the plateaus of φ.

2. Monotone maps of the circle

- Example: $\rho(g) = (\sqrt{5} - 1)/2$





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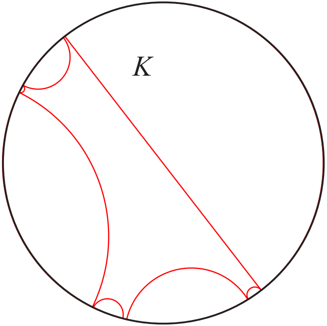

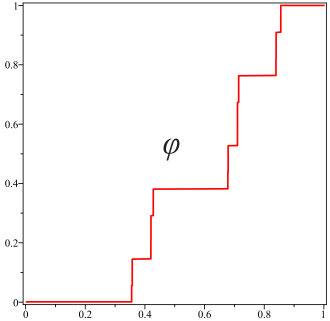

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$$\varphi_* \mu = \lambda.$$

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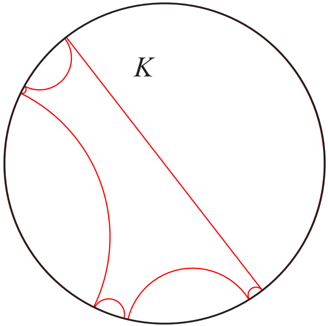

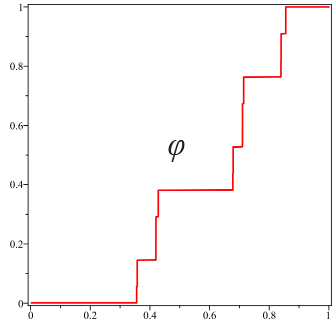

There is a unique g-invariant measure μ supported on K which maps to Lebesgue measure under φ:

$$\varphi_* \mu = \lambda .$$

Similar to the rational case, we have $\varphi(t) = \mu[0, t]$.

3. Rotation sets

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$$m_d(t) = d \cdot t \pmod{\mathbb{Z}}$$

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Definition

A non-empty compact set $X \subset \mathbb{T}$ is a ***rotation set*** for m_d if

- $m_d(X) = X$, and
- the restriction $m_d|_X$ extends to a degree 1 monotone map of the circle.

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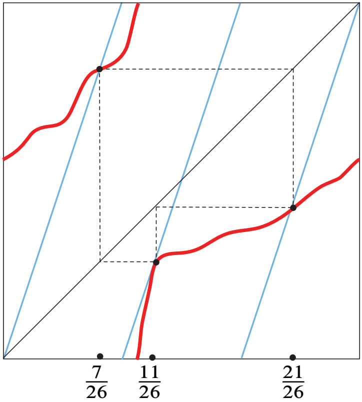
- $m_d(X) = X$, and
- the restriction $m_d|_X$ extends to a degree 1 monotone map of the circle.

Thus, m_d is order-preserving on X, except that it may identify some pairs.

3. Rotation sets

- Example:

$$\frac{7}{26} \overset{m_3}{\longmapsto} \frac{21}{26} \overset{m_3}{\longmapsto} \frac{11}{26}$$



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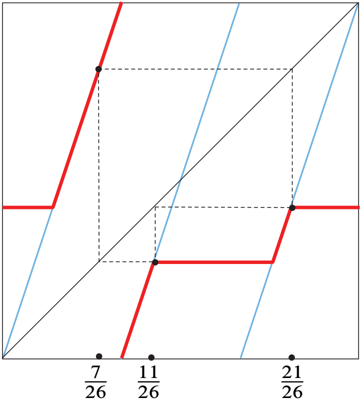

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Every rotation set is nowhere dense, whereas a randomly chosen point on the circle has a dense orbit under m_d.

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Theorem

The union of all rotation sets for m_d has Lebesgue measure zero.

3. Rotation sets

Let X be a rotation set for m_d.

Definition

The ***rotation number*** $\rho(X) \in [0, 1)$ is defined as the rotation number of any degree 1 monotone extension g of $m_d|_X$.

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Definition

The ***rotation number*** $\rho(X) \in [0, 1)$ is defined as the rotation number of any degree 1 monotone extension g of $m_d|_X$.

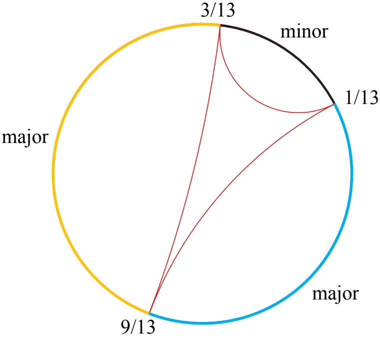
- $\rho(X) = p/q$ in lowest terms iff X has a q-cycle under m_d.

4. Gaps and their dynamics

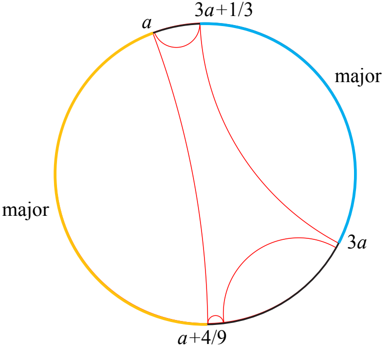
Definition

- A connected component of $\mathbb{T} \setminus X$ is called a ***gap*** of X.
- A gap of length ℓ is ***minor*** if $\ell < 1/d$, and ***major*** otherwise.
- A major gap is ***taut*** if $d \cdot \ell$ is an integer, and ***loose*** otherwise.
- The ***multiplicity*** of a major gap is the integer part of $d \cdot \ell$.

4. Gaps and their dynamics



$d = 3, \rho = 1/3$



$d = 3, \rho = (\sqrt{5} - 1)/2$

4. Gaps and their dynamics

Suppose X is not a single (fixed) point. Define the ***standard monotone map*** g as follows:

On a minor gap, set $g = m_d$.

On a major gap $(a, a + \ell)$ of multiplicity n, set

$$g(t) = \begin{cases} m_d(a) & t \in (a, a + n/d] \\ m_d(t) & t \in (a + n/d, a + \ell). \end{cases}$$

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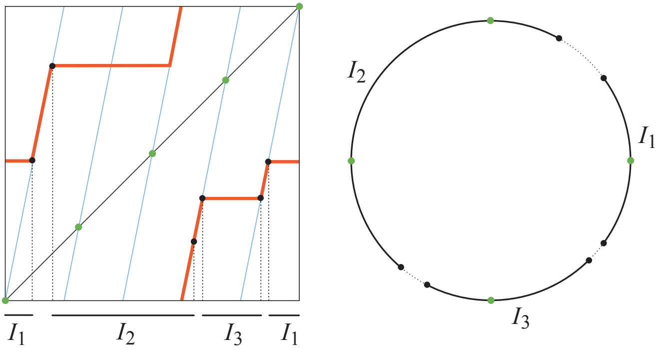

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Theorem

Suppose X is not a single point and I is a gap of length ℓ.

(i) If I is minor, the image $g(I)$ is a gap of length $d\cdot\ell$.

(ii) If I is taut, $g(I)$ is a single point in X.

(iii) If I is loose, $g(I)$ is a gap of length $\{d\cdot\ell\}$.

4. Gaps and their dynamics

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Suppose X is not a single point and I is a gap of X. Then either I is periodic or it eventually maps to a taut gap.

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Corollary

If $\rho(X)$ is irrational, every gap of X eventually maps to a taut gap. In particular, at least one major gap of X is taut.

5. Minimal rotation sets

A minimal rational rotation set is a cycle.

Theorem

Every rotation set X for m_d with $\rho(X) = p/q$ contains finitely many cycles $C_1, \ldots, C_N$ where $1 \leq N \leq d-1$. Moreover,

(i) Each C_i is a q-cycle with combinatorial rotation number p/q.

(ii) For $i \neq j$ the cycles C_i and C_j are "superlinked."

(iii) $X \smallsetminus (C_1 \cup \cdots \cup C_N)$ is at most countable, with every point eventually mapping to $C_1 \cup \cdots \cup C_N$ under the iterations of m_d.

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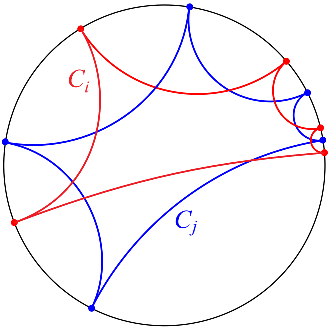

5. Minimal rotation sets

• Example: Under the tripling map m_3 there are five 4-cycles of rotation number $1/4$:

$$C_1 : \frac{1}{80} \mapsto \frac{3}{80} \mapsto \frac{9}{80} \mapsto \frac{27}{80}$$
$$C_2 : \frac{2}{80} \mapsto \frac{6}{80} \mapsto \frac{18}{80} \mapsto \frac{54}{80}$$
$$C_3 : \frac{5}{80} \mapsto \frac{15}{80} \mapsto \frac{45}{80} \mapsto \frac{55}{80}$$
$$C_4 : \frac{14}{80} \mapsto \frac{42}{80} \mapsto \frac{46}{80} \mapsto \frac{58}{80}$$
$$C_5 : \frac{41}{80} \mapsto \frac{43}{80} \mapsto \frac{49}{80} \mapsto \frac{67}{80}$$

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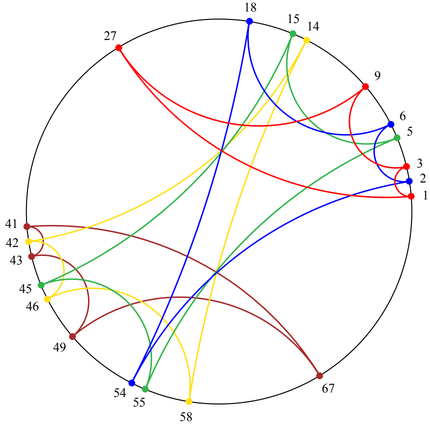
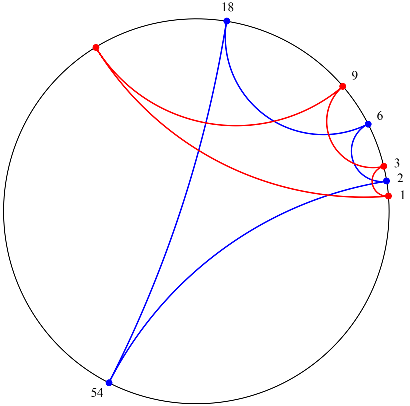


5. Minimal rotation sets

But only four unions of superlinked pairs form rotation sets:

$C_1 \cup C_2$



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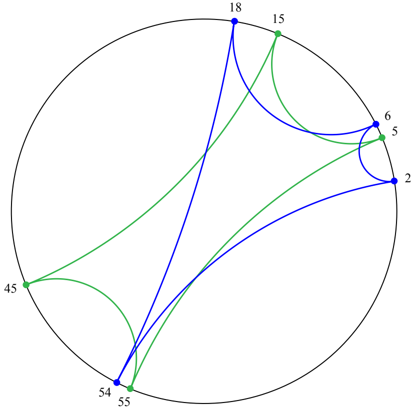

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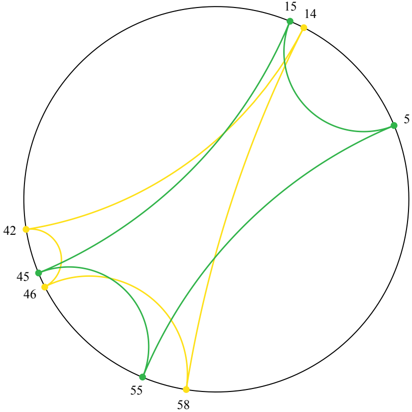

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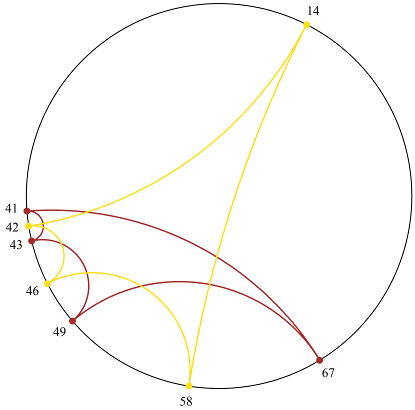

5. Minimal rotation sets

Now consider the irrational case.

Theorem

Every irrational rotation set X for m_d contains a unique minimal rotation set K. Moreover,

(i) K is the Cantor attractor of any monotone extension of $m_d|_X$.

(ii) Each gap of K contains at most finitely many points of X, all of which eventually map to K under the iterations of m_d.

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Now consider the irrational case.

Theorem

Every irrational rotation set X for m_d contains a unique minimal rotation set K. Moreover,

(i) *K is the Cantor attractor of any monotone extension of $m_d|_X$.*

(ii) *Each gap of K contains at most finitely many points of X, all of which eventually map to K under the iterations of m_d.*

Corollary

Suppose X is a minimal rotation set for m_d with $\rho(X) = \theta$ irrational. Then there exists a degree 1 monotone map $\varphi : \mathbb{T} \to \mathbb{T}$, whose plateaus are precisely the gaps of X, which satisfies $\varphi \circ m_d = r_\theta \circ \varphi$ on X.