Mating the Basilica with a Siegel Disc

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Topics in Complex Dynamics, 2016
Universitat de Barcelona

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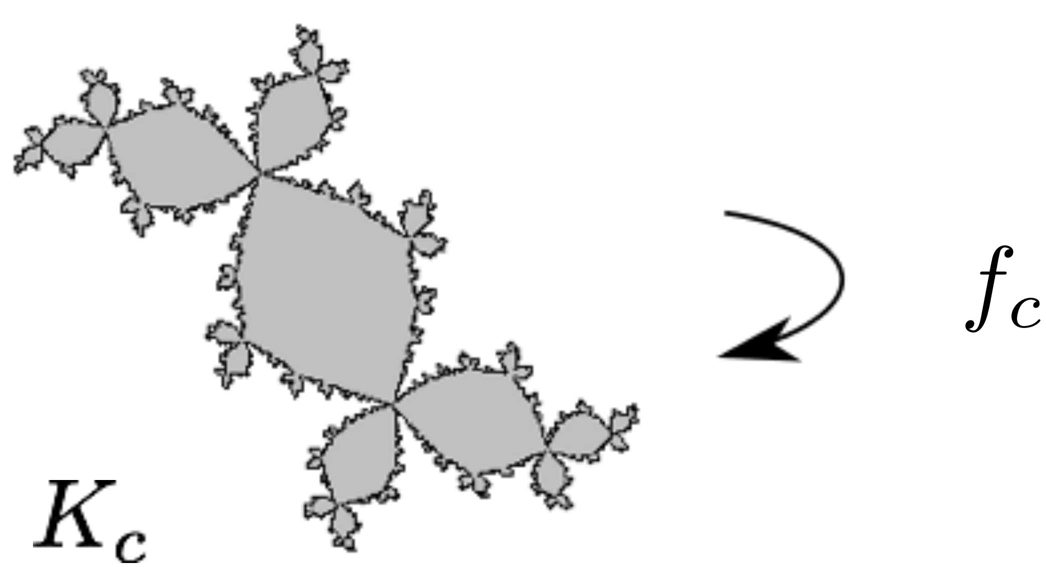

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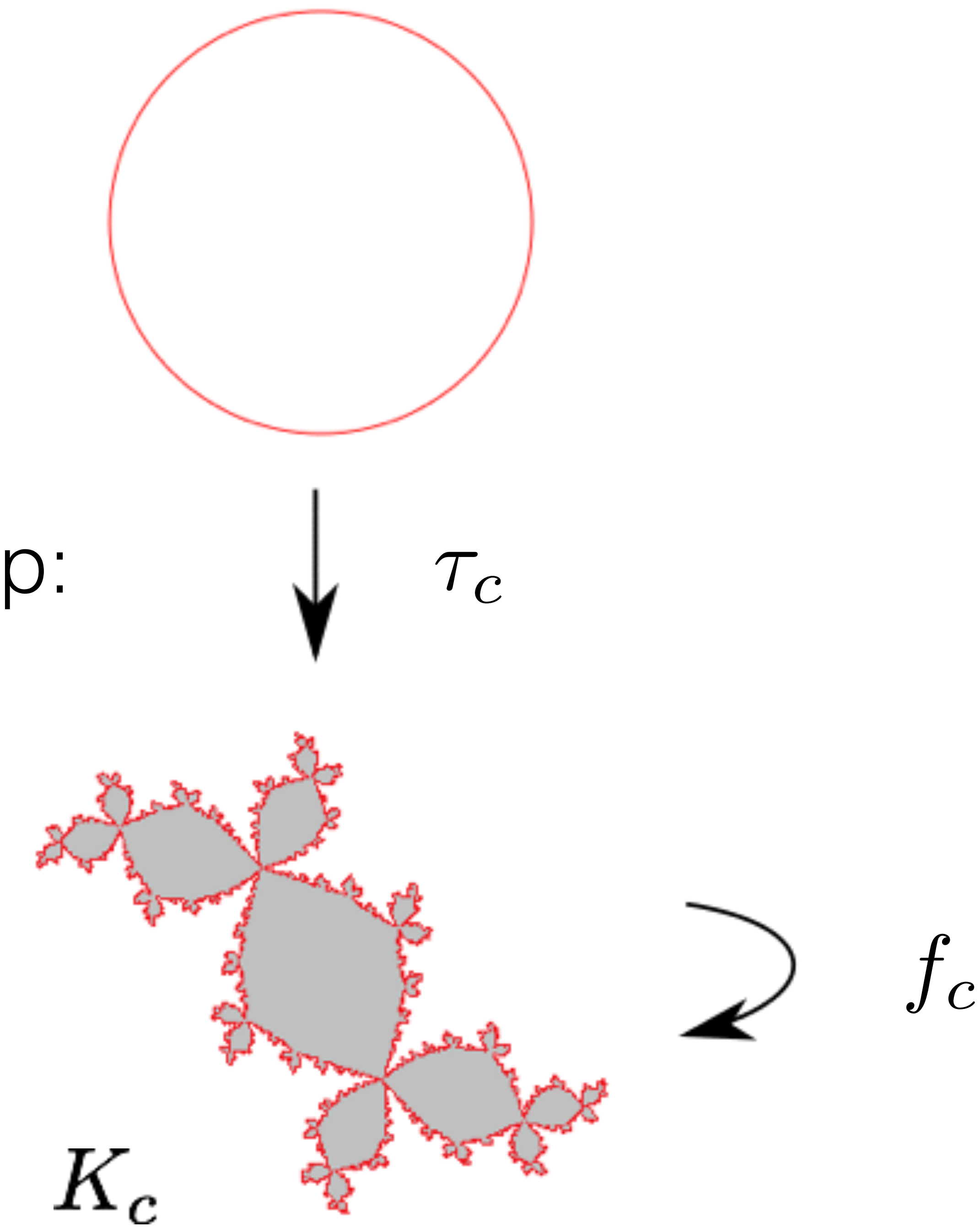

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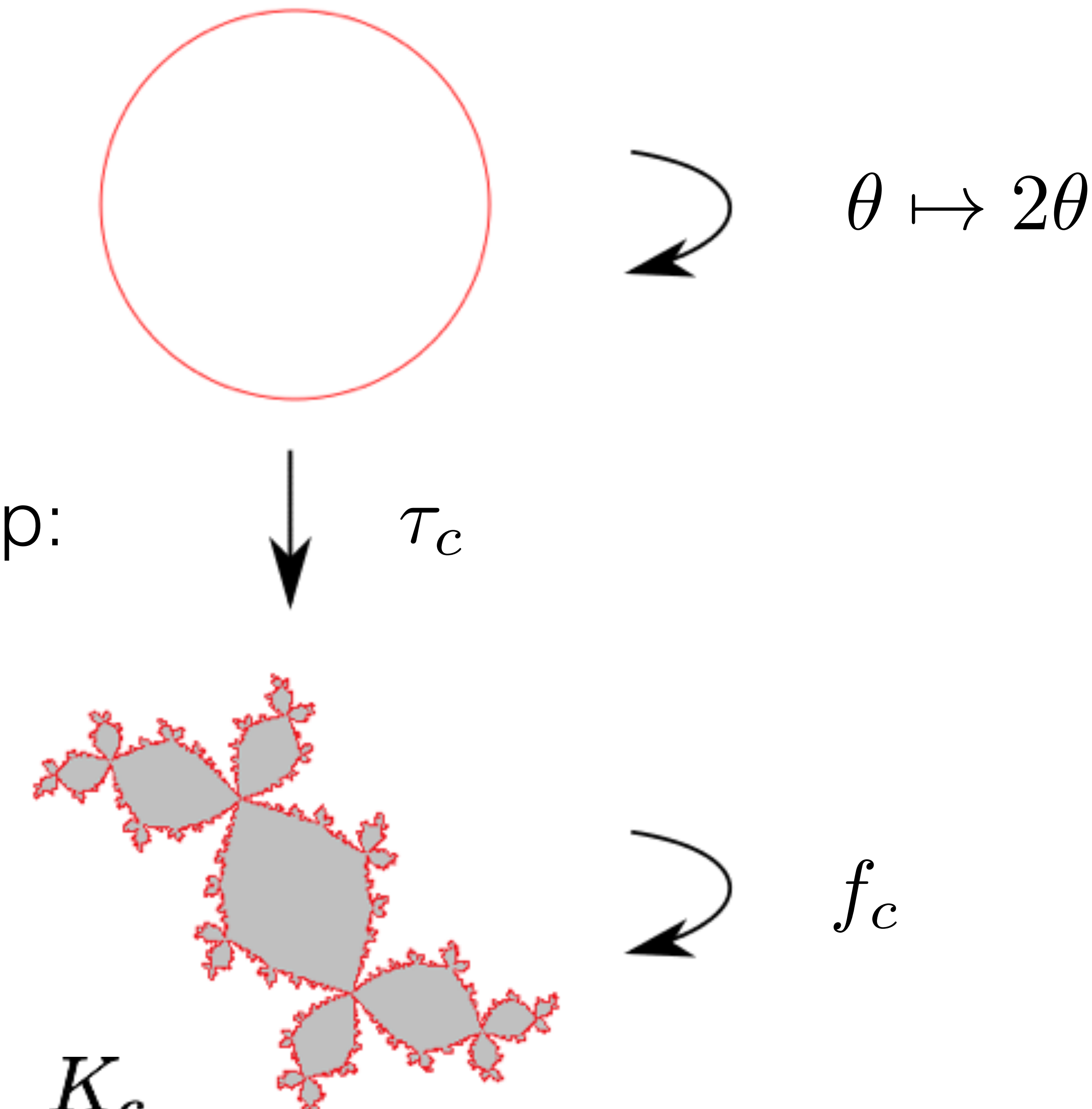

Mating Construction [Douady, Hubbard]

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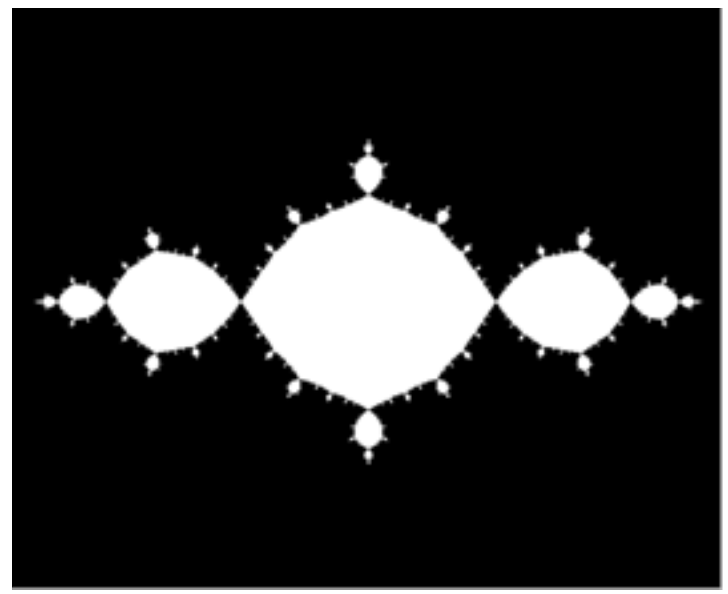
K_c K_d



f_c f_d

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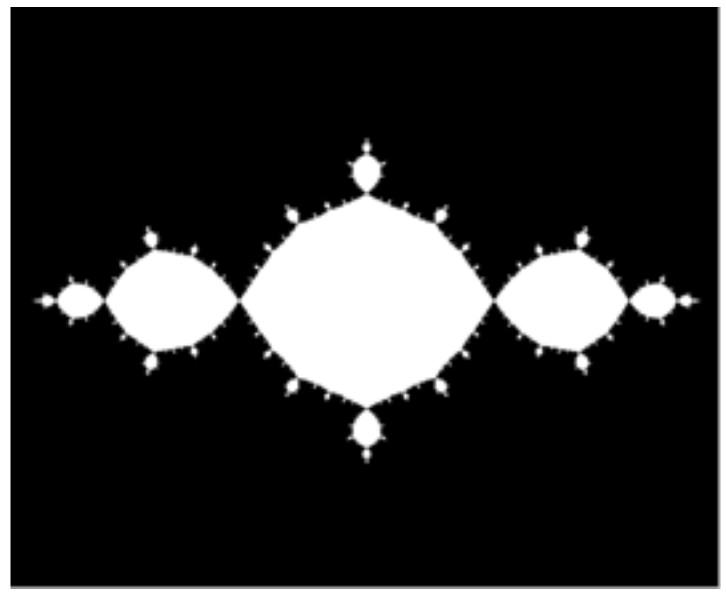
$\vee$



f_c f_d

$$K_c \vee K_d = (K_c \sqcup K_d)/\{\tau_c(\theta) \sim \tau_d(-\theta)\}$$

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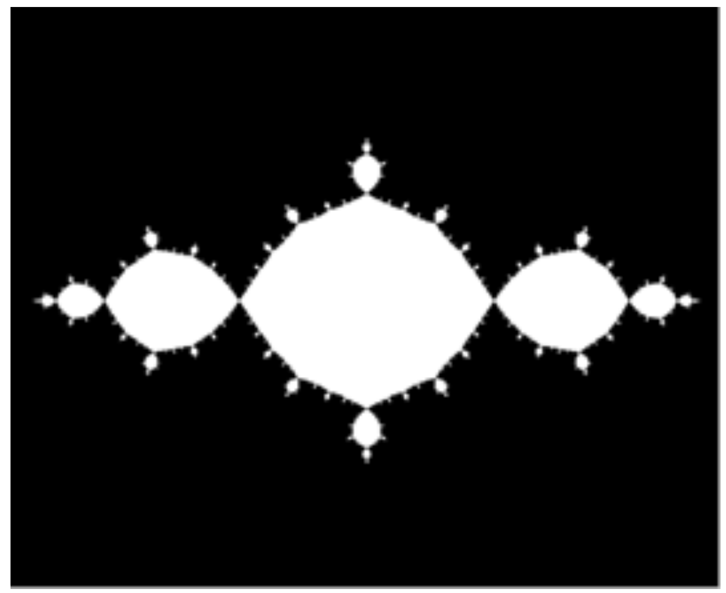
K_c $\quad$ K_d $\quad$ $K_c \vee K_d$

$\bigvee$



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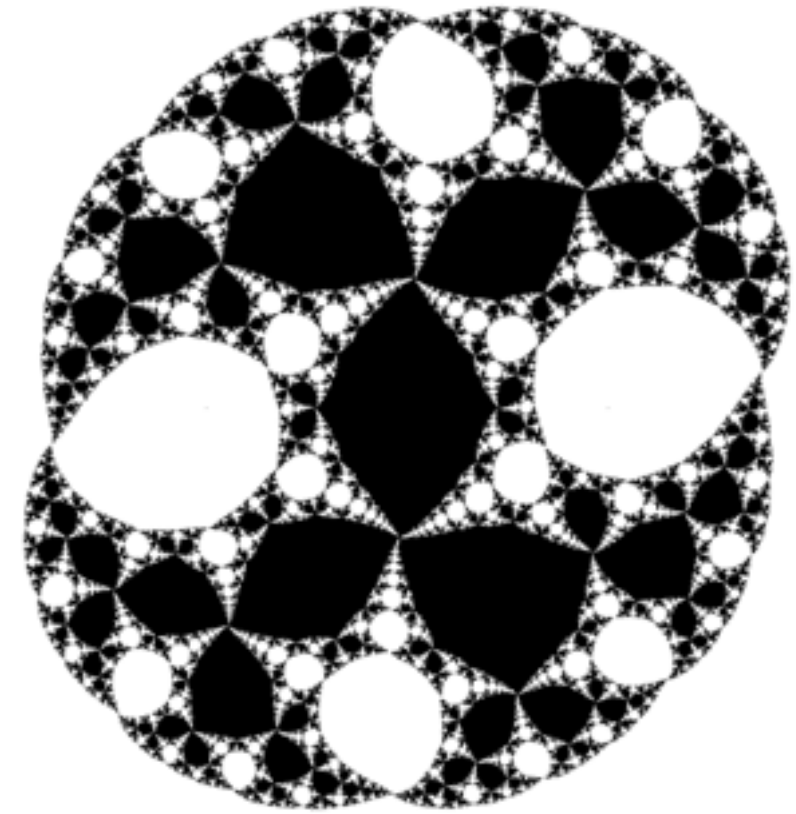

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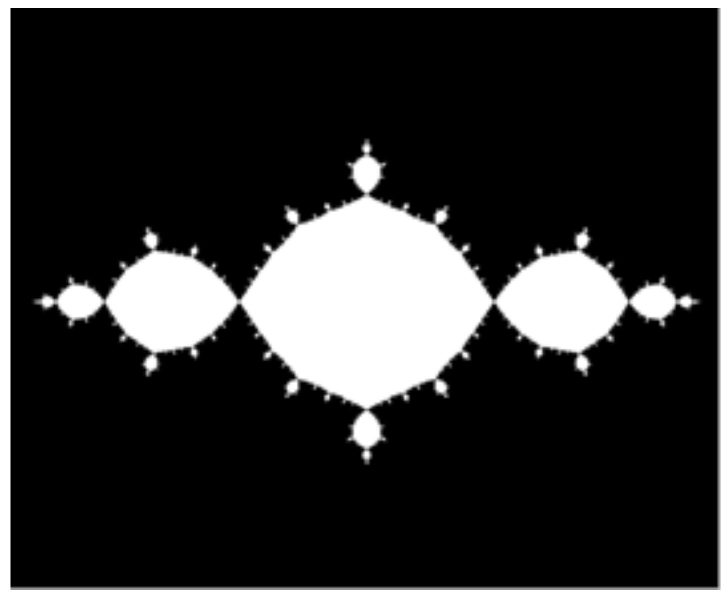

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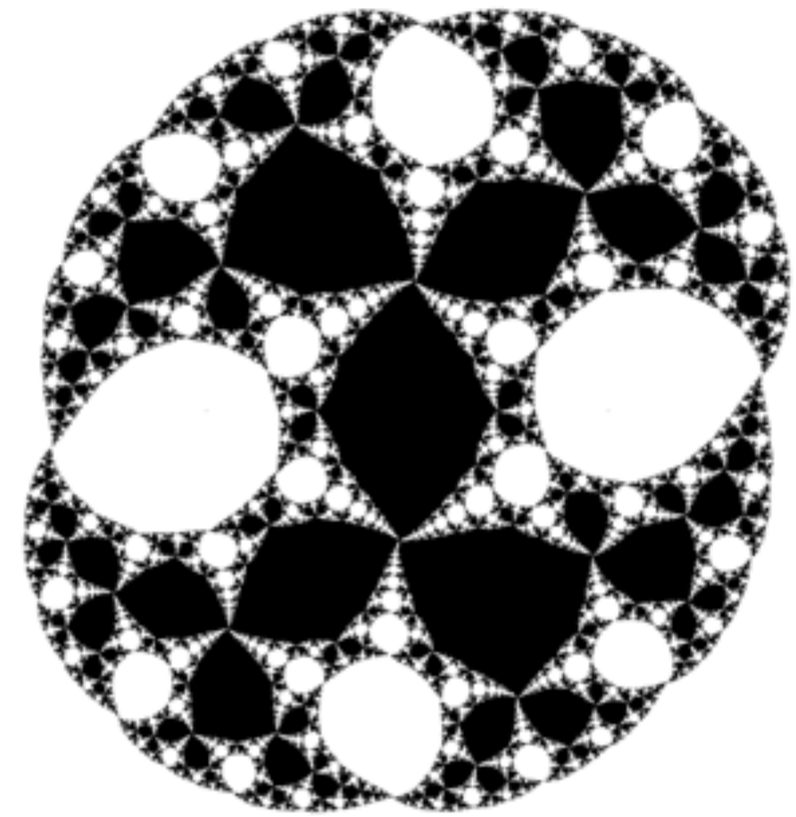

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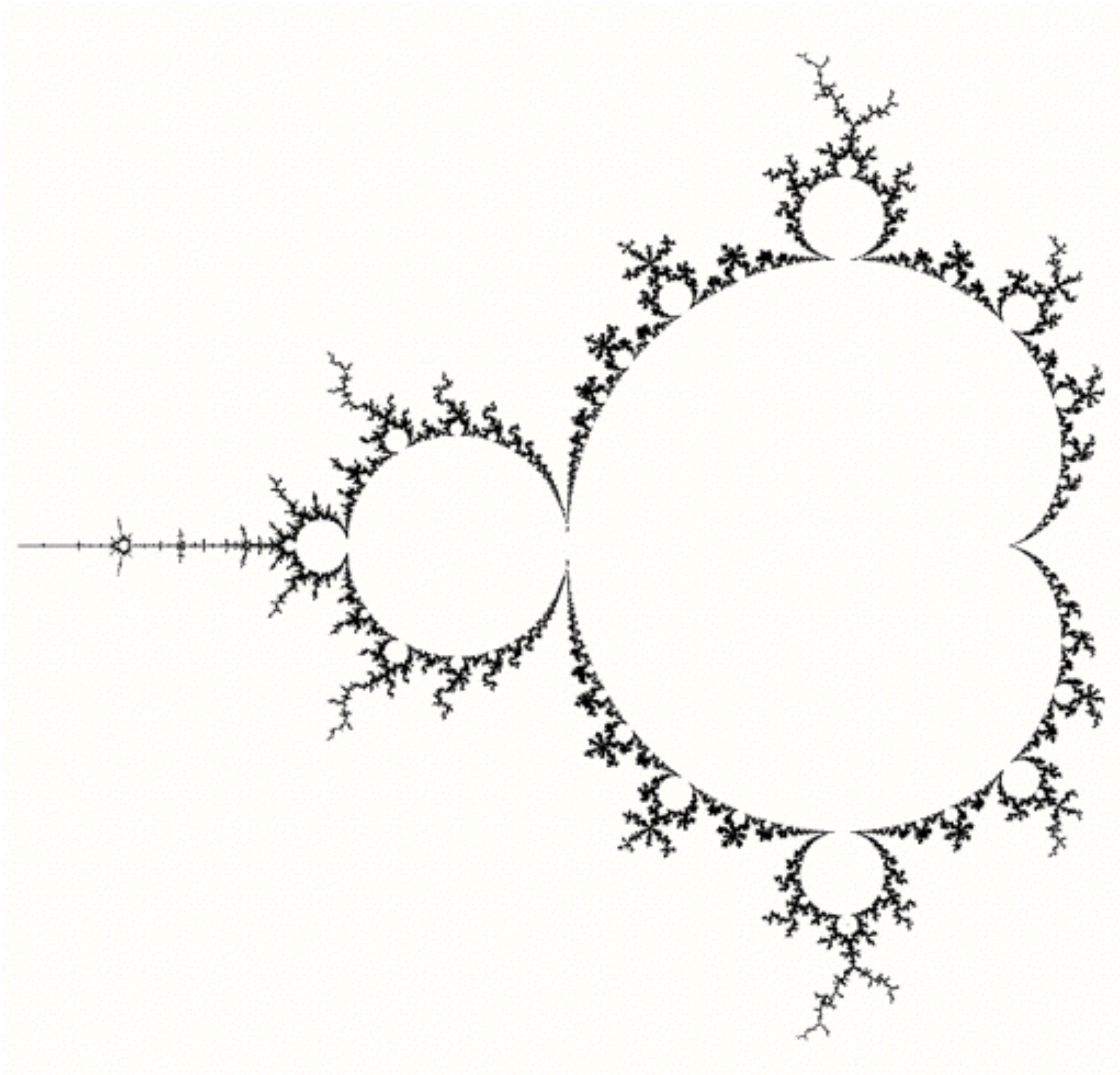
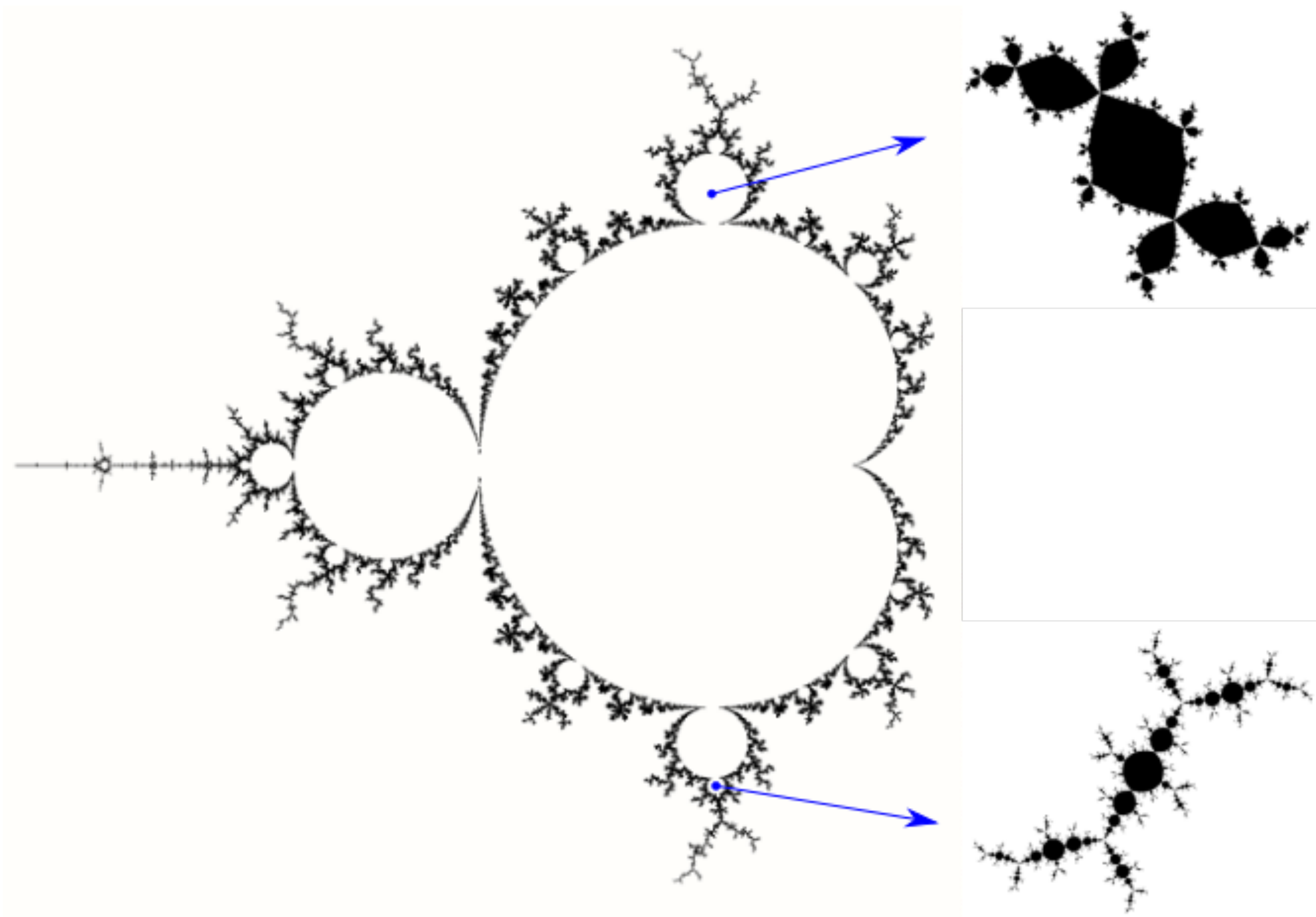
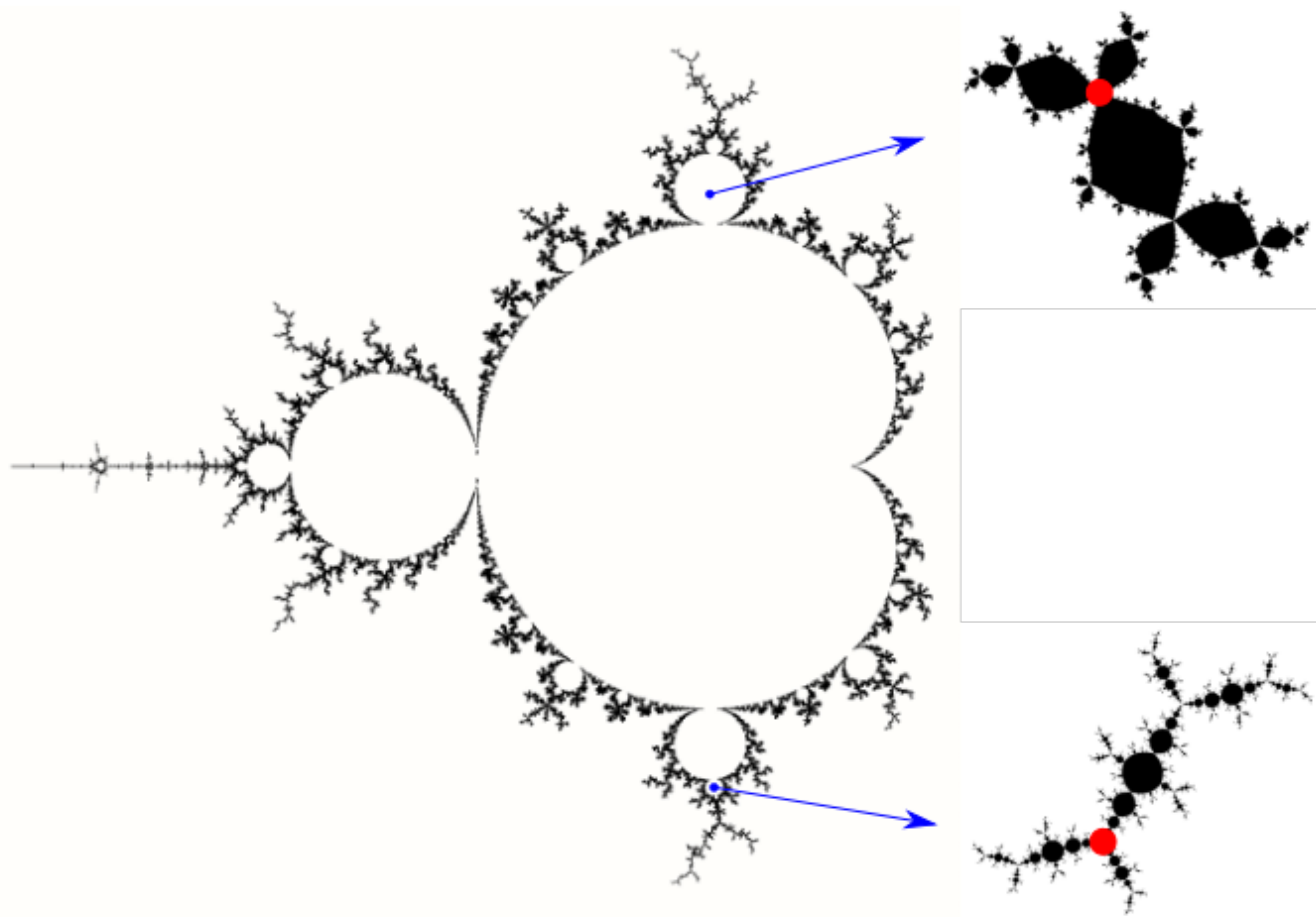
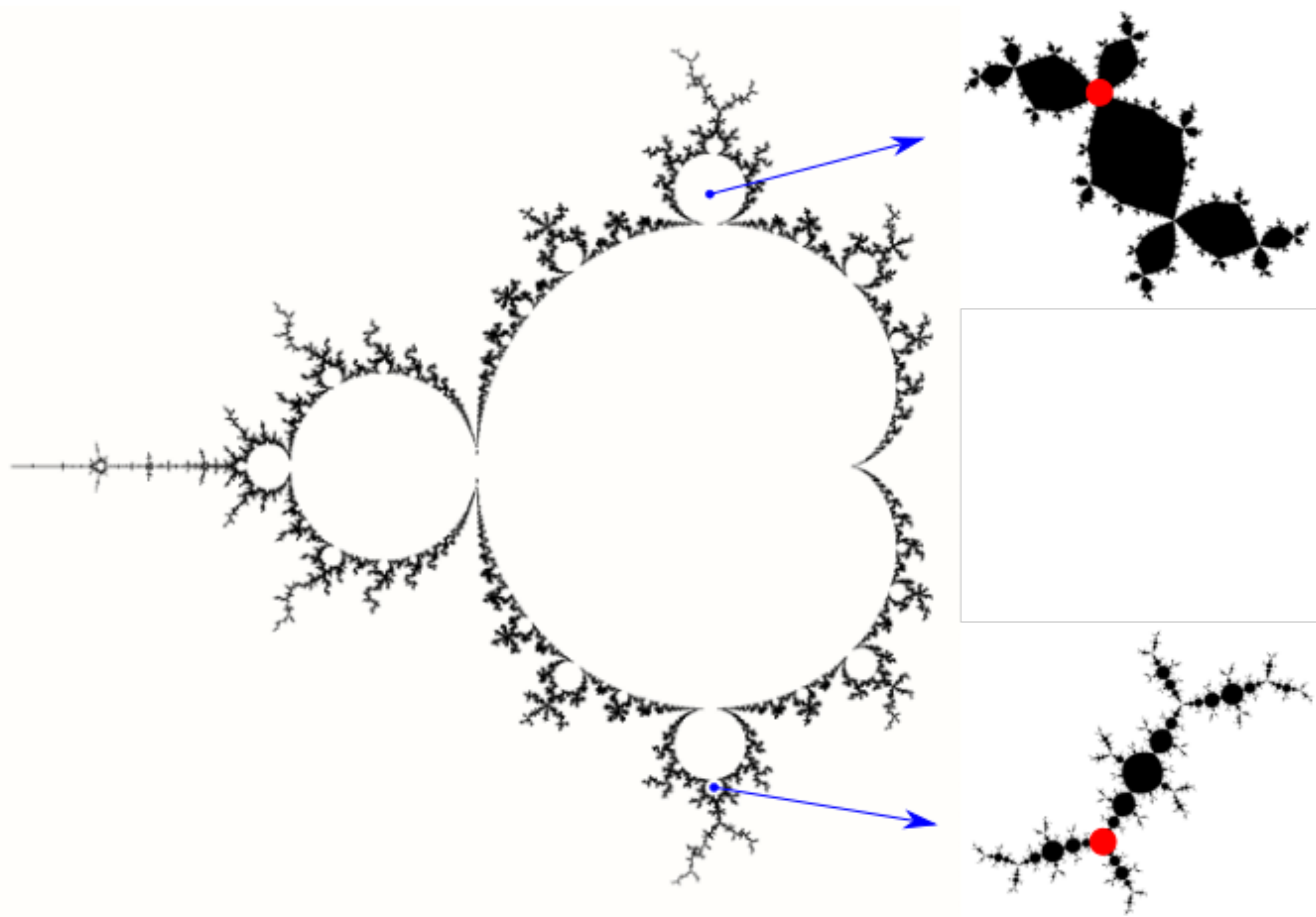
If $f_c \vee f_d$ can be realized by a rational map, we say that f_c and f_d are **mateable**.









[Rees, Tan, Shishikura] Suppose f_c and f_d are post-critically finite. Then f_c and f_d are mateable if and only if c and d do not belong in conjugate limbs.

The Basilica Family

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Consider $R_a(z) = \dfrac{a}{z^2 + 2z}$, $a \in \mathbb{C} \setminus \{0\}$.

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-1 is a free critical point, and $-a$ is a free critical value.

The Basilica Polynomial

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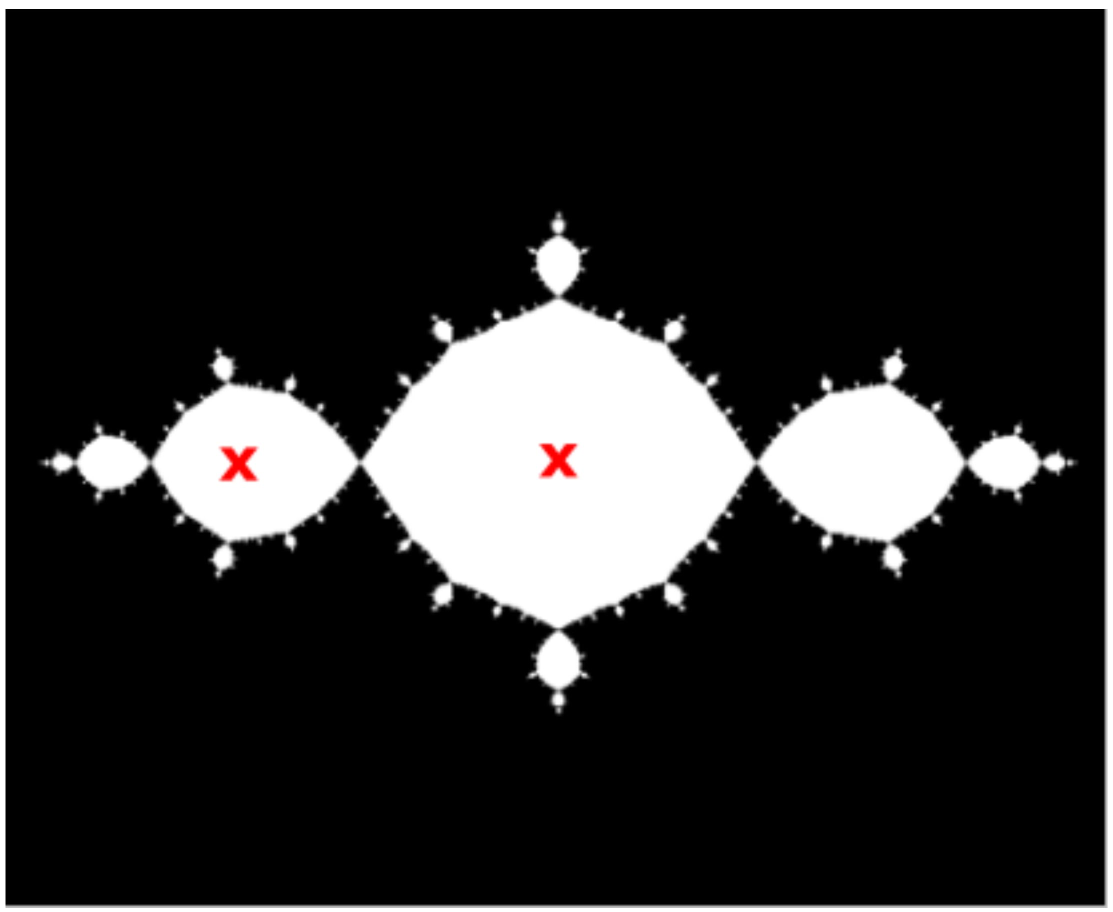
$$f_B(z) := z^2 - 1$$



$$\{0, -1\}, \{\infty\}$$

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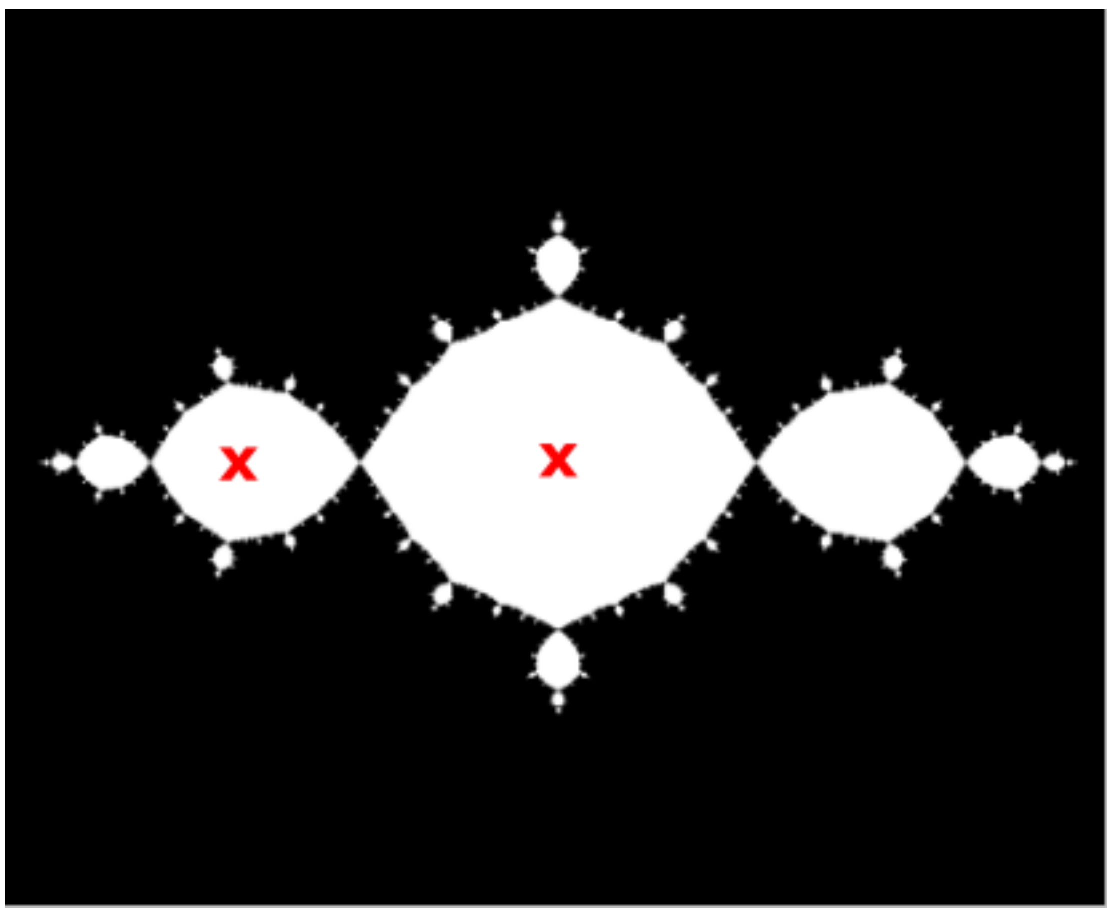

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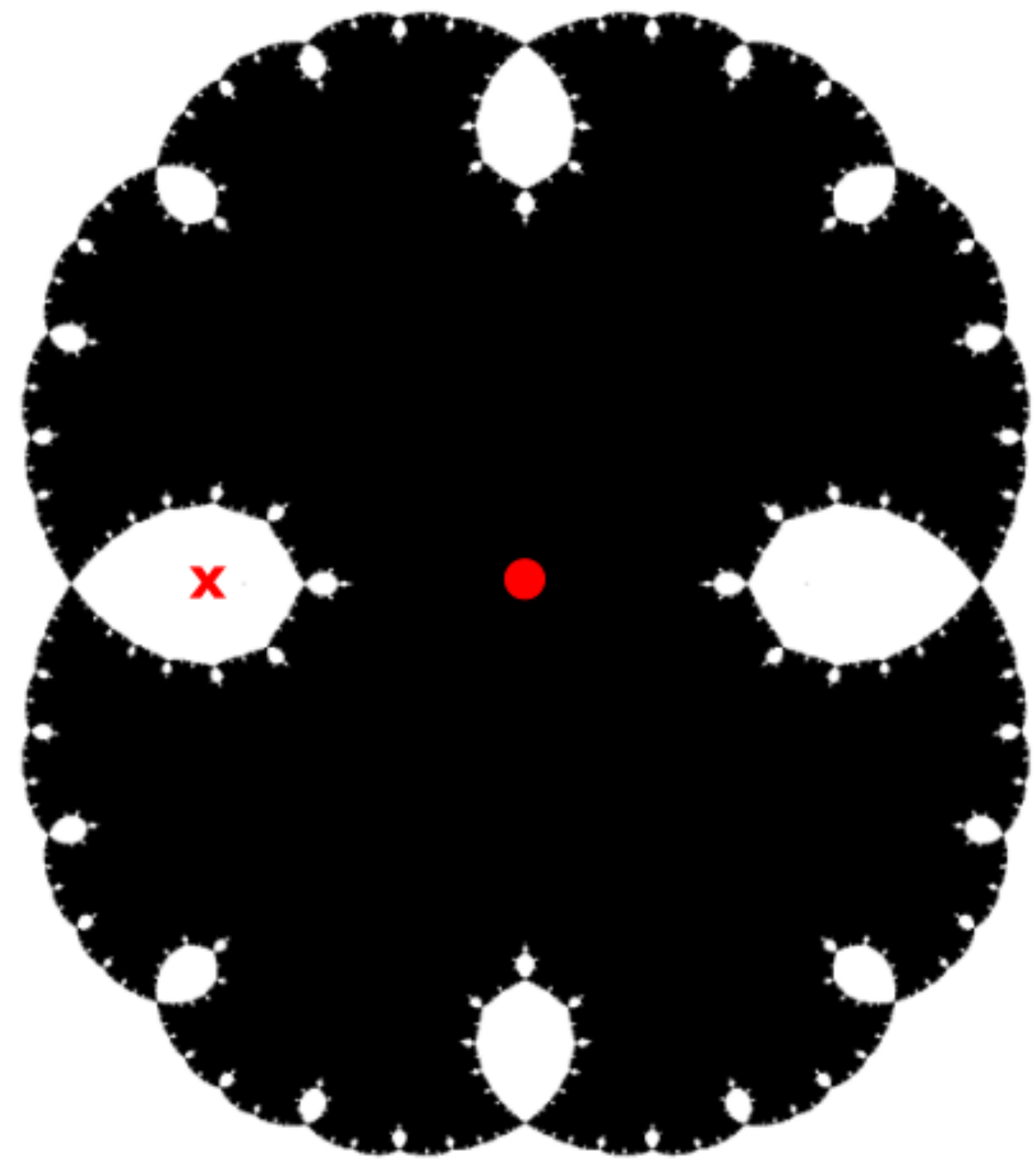

$\{0, -1\}, \{\infty\}$

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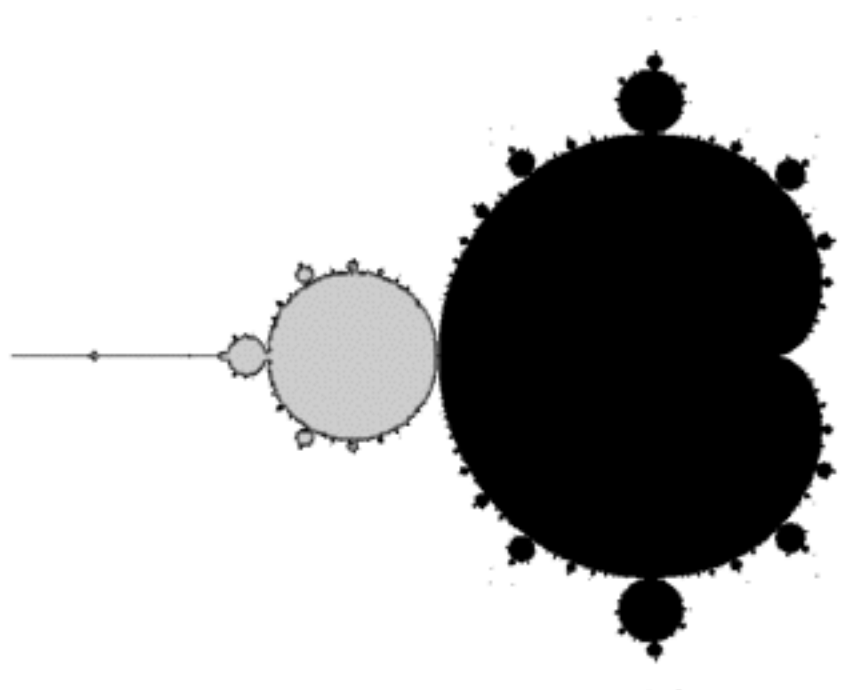
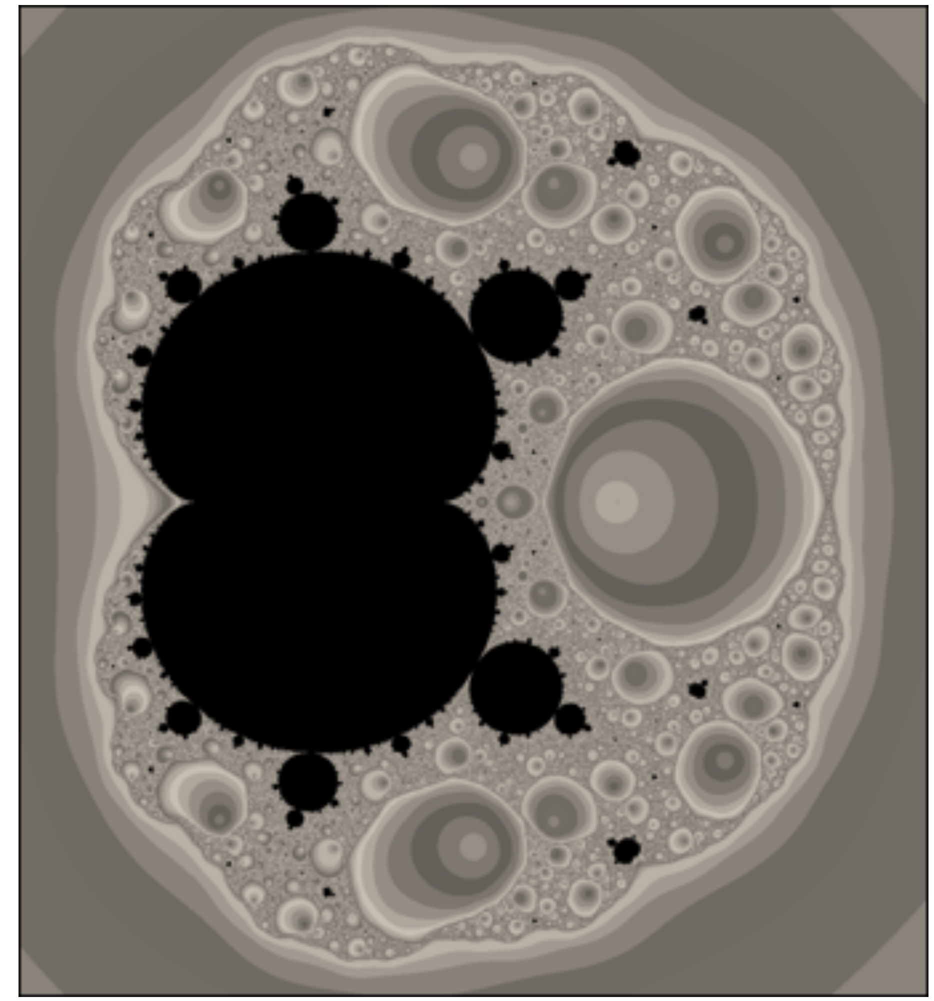
c-plane



$$\mathcal{M} = \{c \mid 0 \notin \mathcal{A}_c^\infty\}$$

a-plane



$$\mathcal{M}_B := \{a \mid -1 \notin \mathcal{B}_a^\infty\}$$

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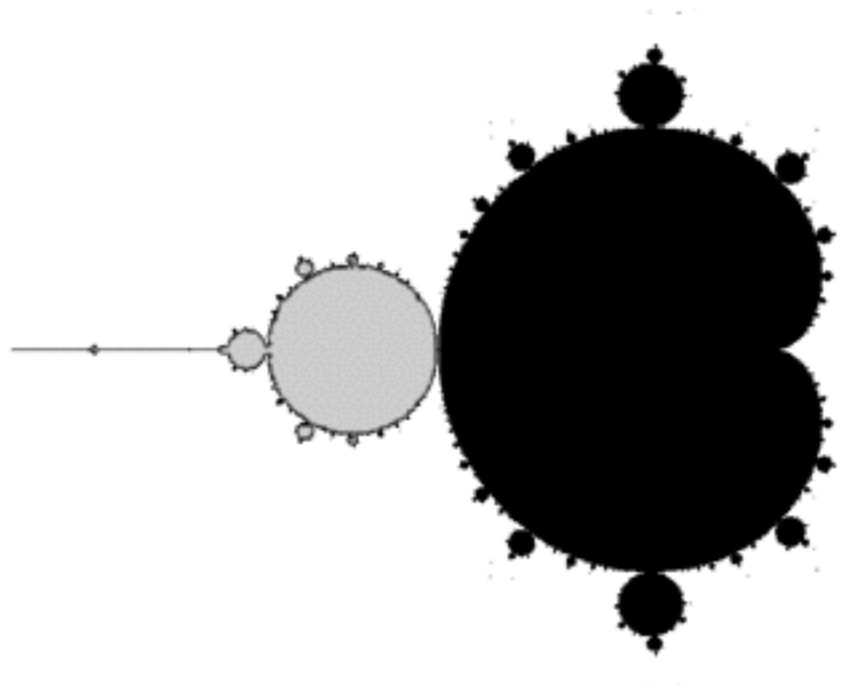

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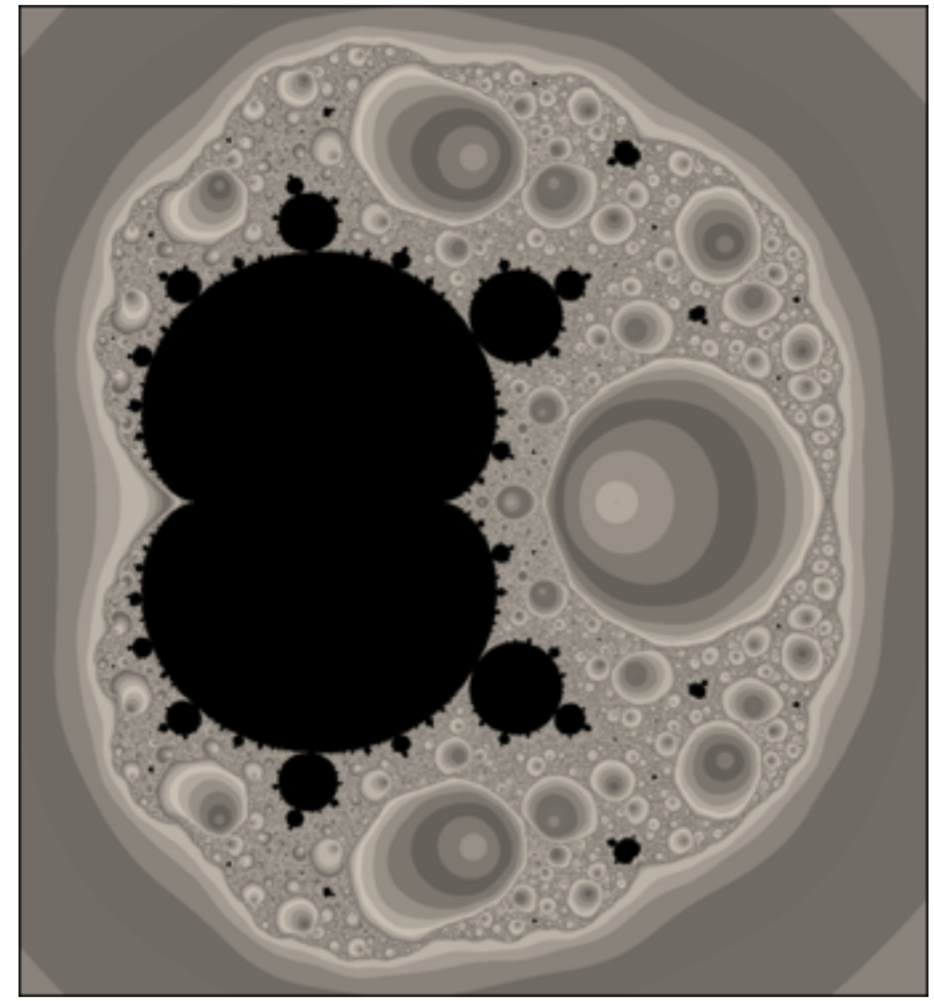

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Can the basilica family be understood as the set of matings of the quadratic family with the basilica polynomial?

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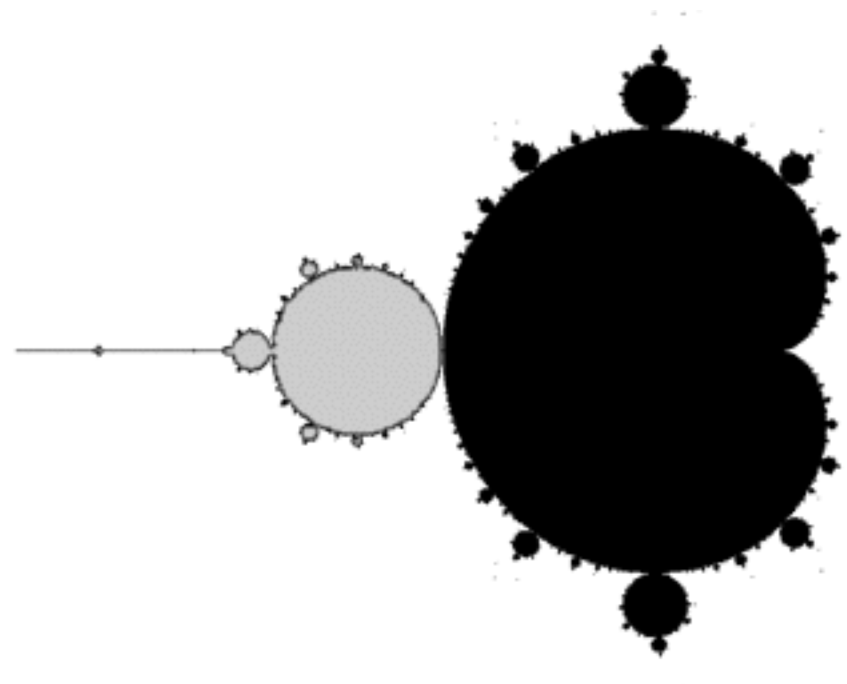

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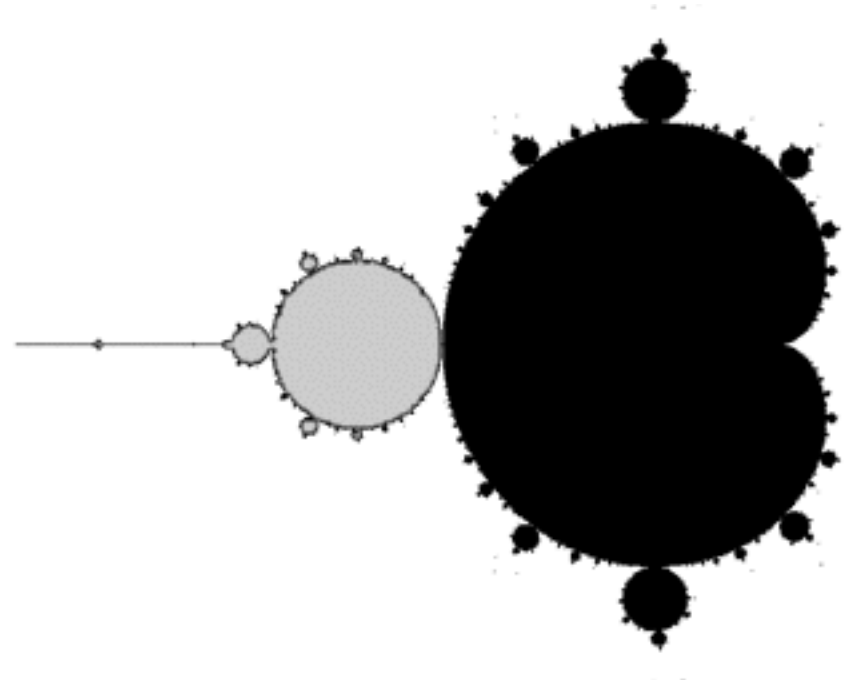

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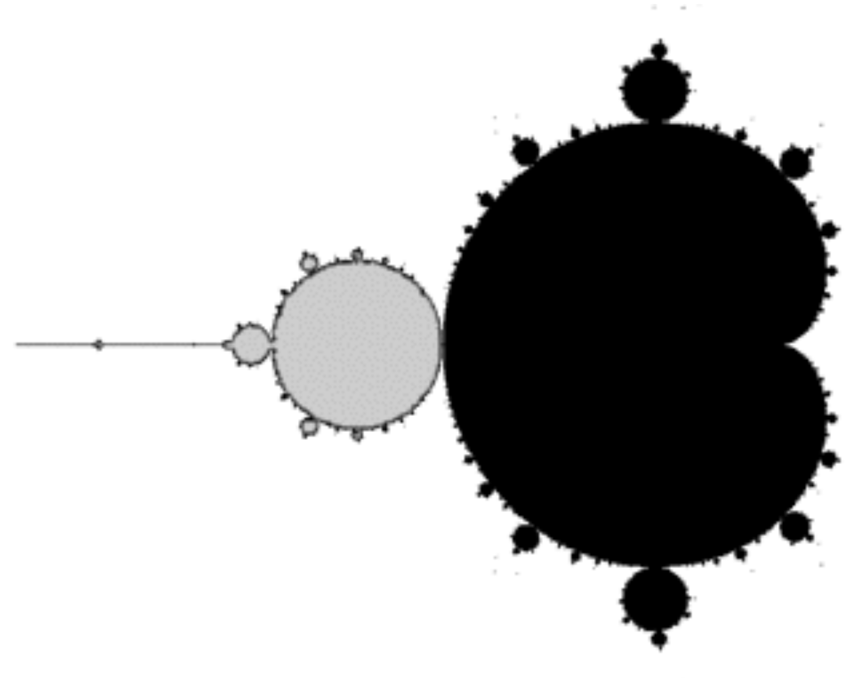

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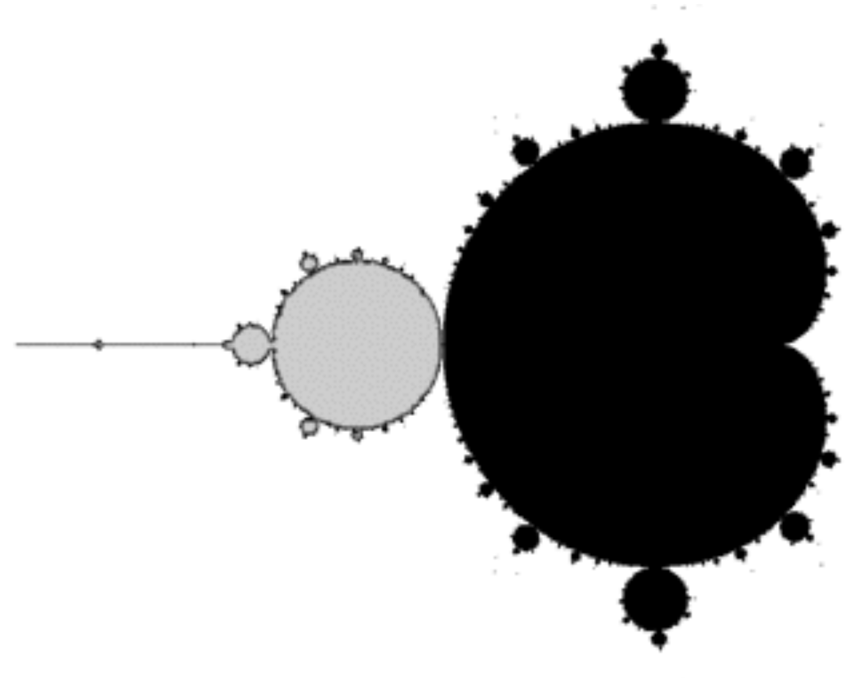

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[Aspenberg, Yampolsky] If f_c is finitely renormalizable, and has no non-repelling periodic orbits, then it is mateable with f_B.

[D. Dudko] If f_c is at least 4 times renormalizable, then it is mateable with f_B.

Boundary of Hyperbolic Components

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If f_c is ***Cremer***, then its Julia set is non-locally connected. Hence it is non-mateable with f_B.

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[Petersen, Zakeri] Suppose f_c has an indifferent periodic orbit $\mathcal{O}$ with rotation number $\nu \notin \mathbb{Q}$. Then for or a.e. ν, $\mathcal{O}$ is Siegel, and f_c has a locally connected Julia set.

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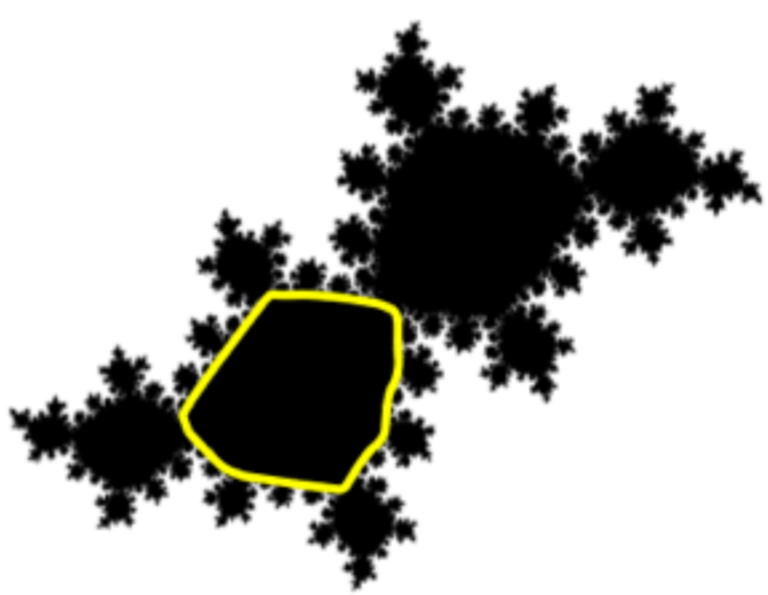
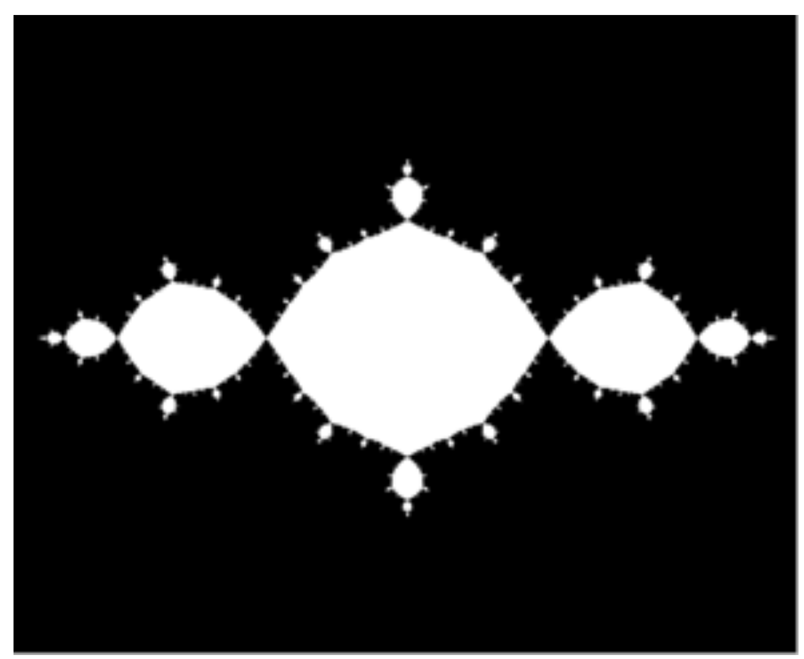


f_S



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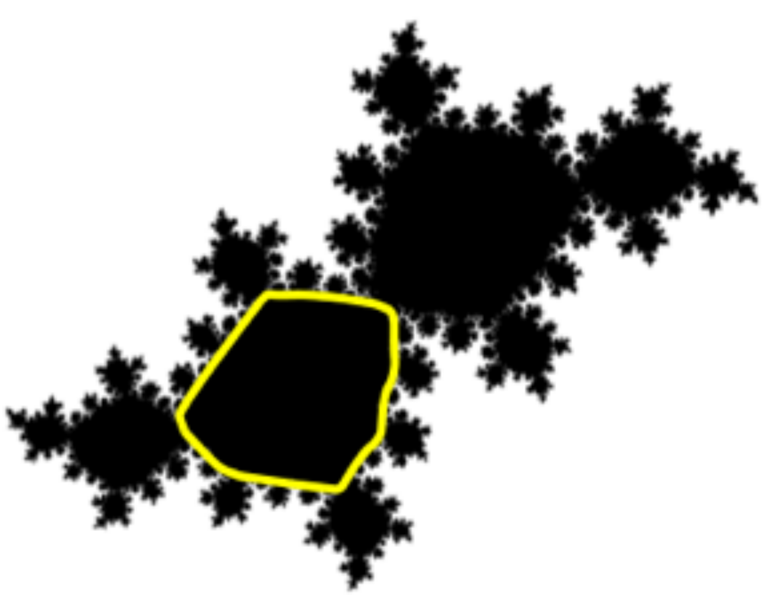

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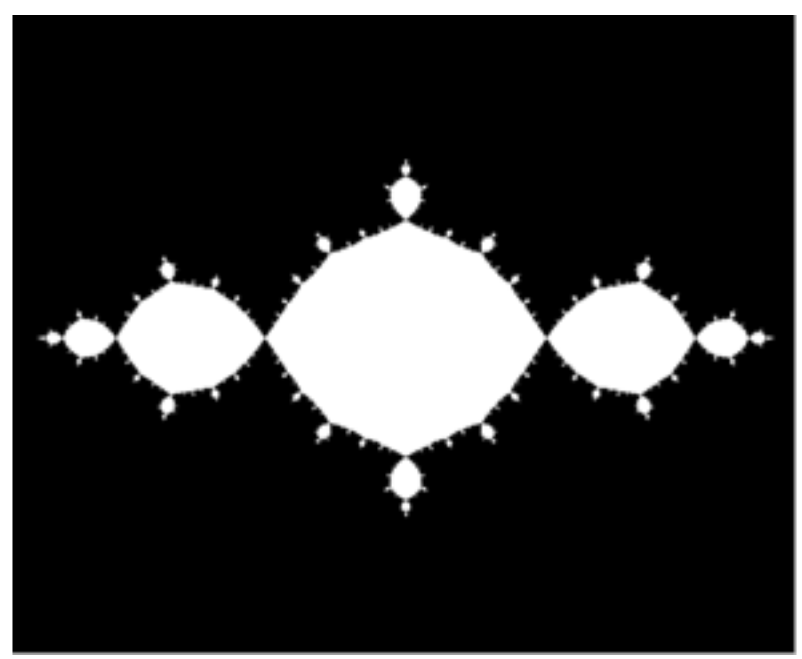
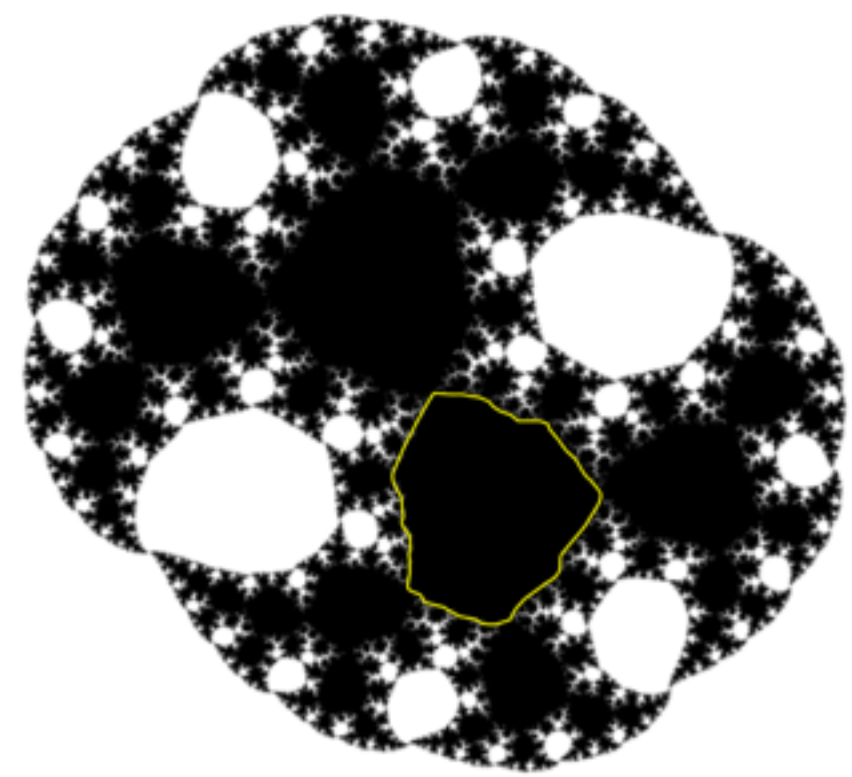
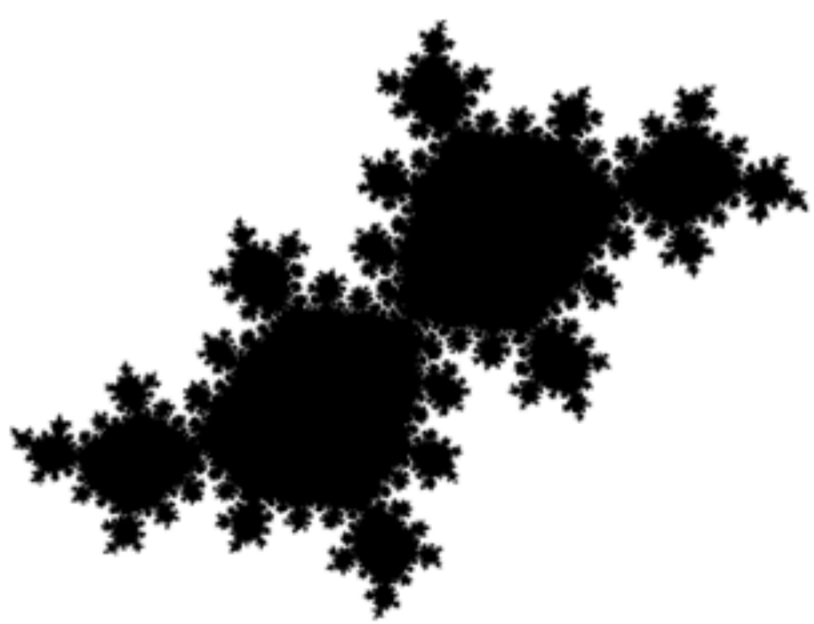
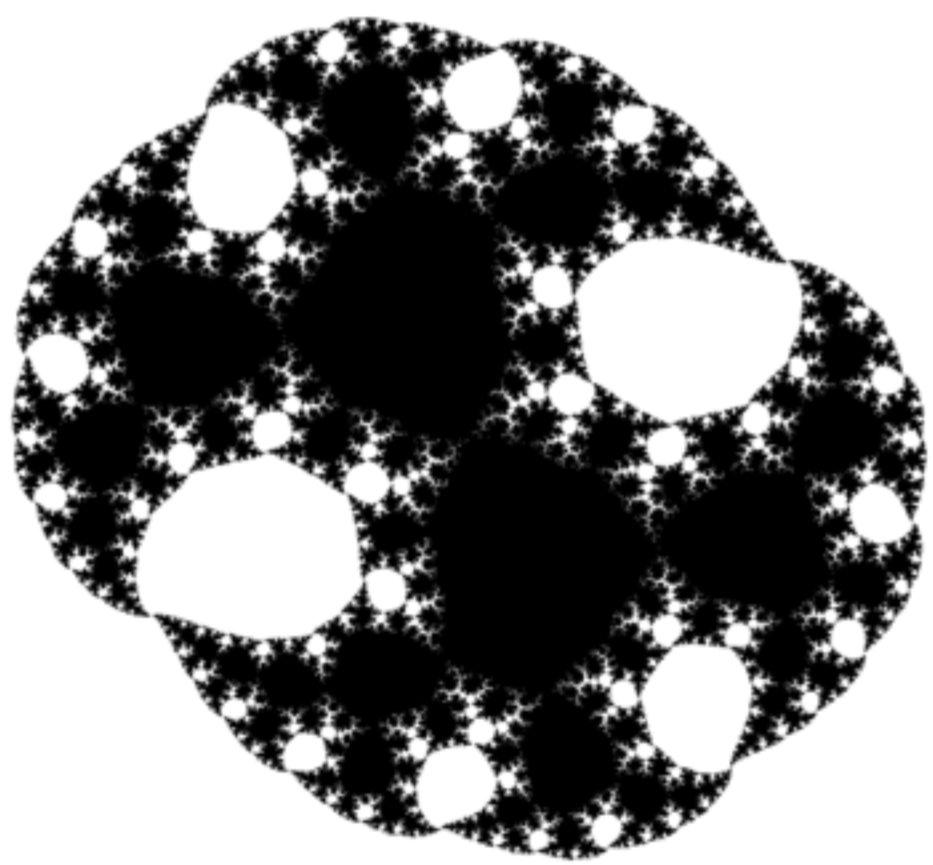


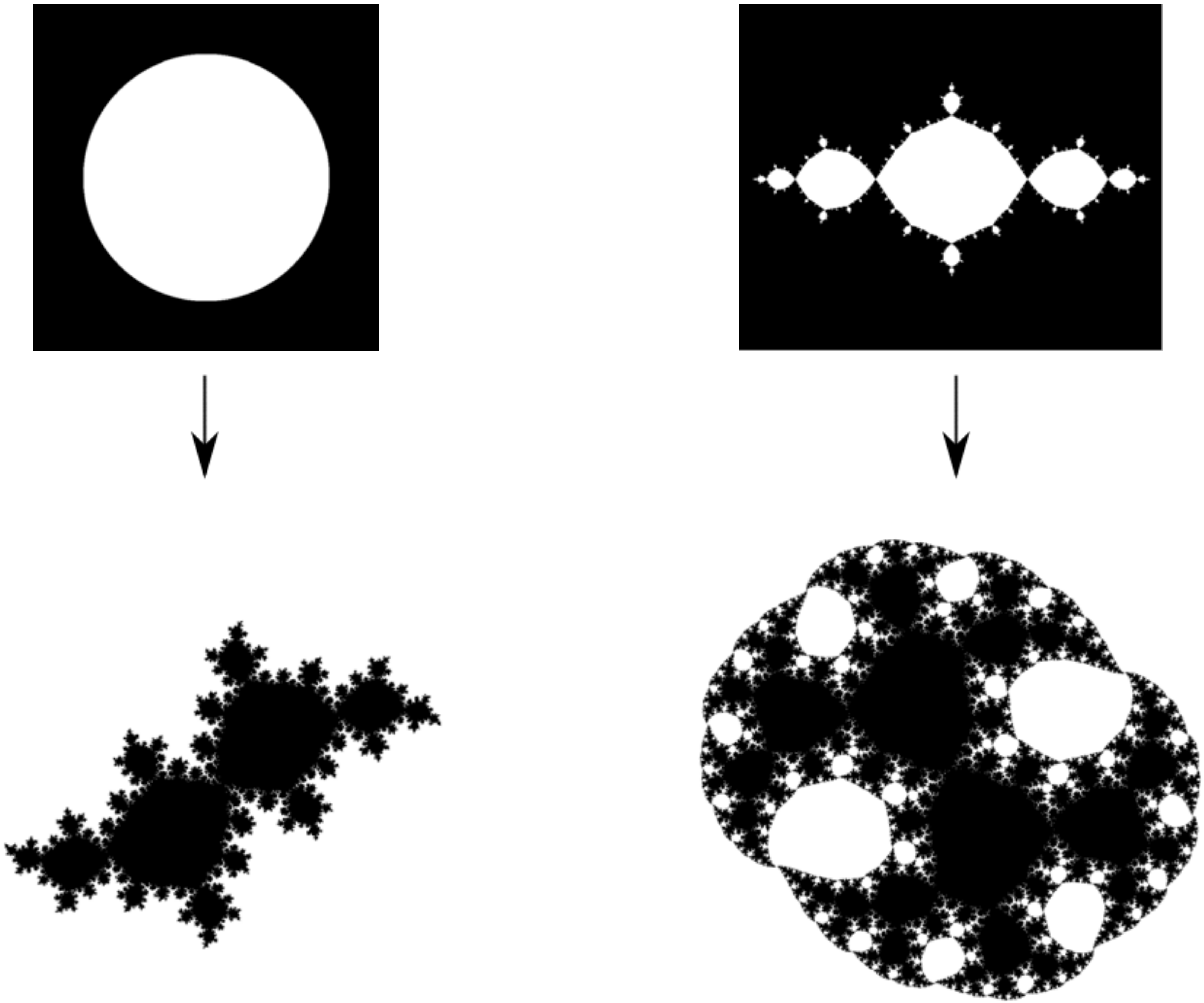
f_B

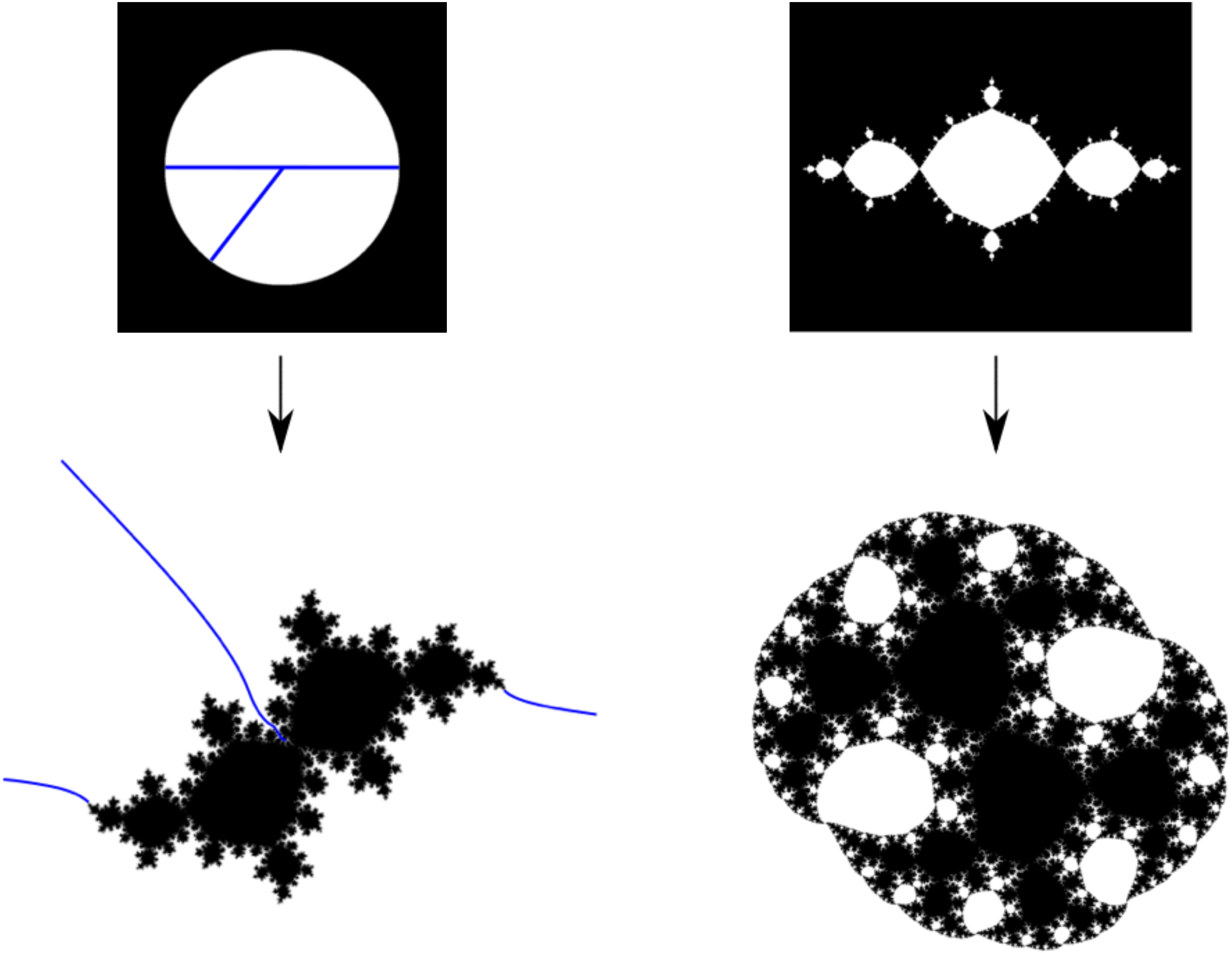


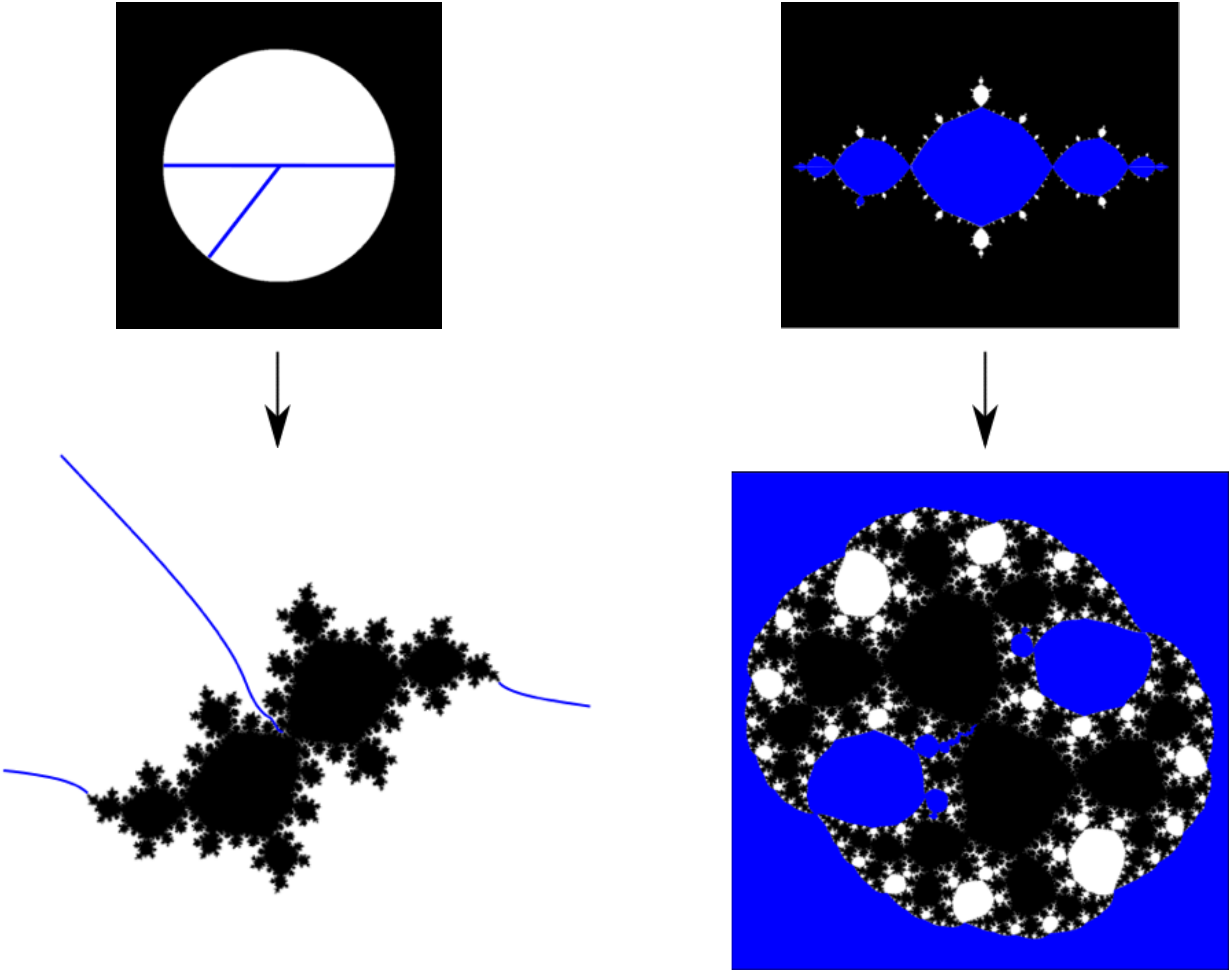
$f_S \vee f_B$



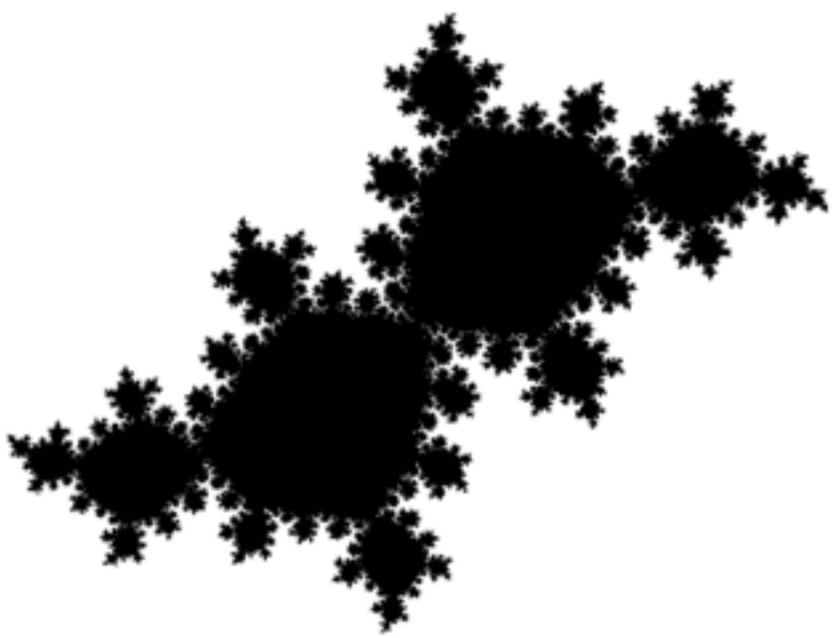


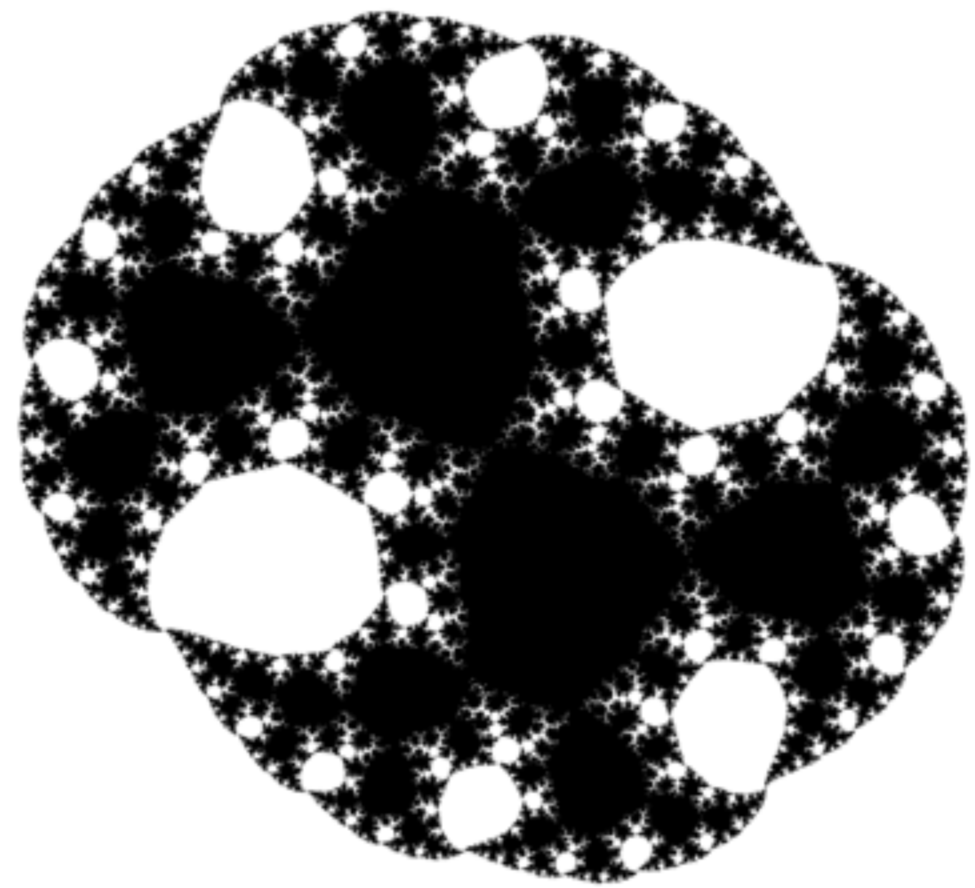






Puzzle Partition





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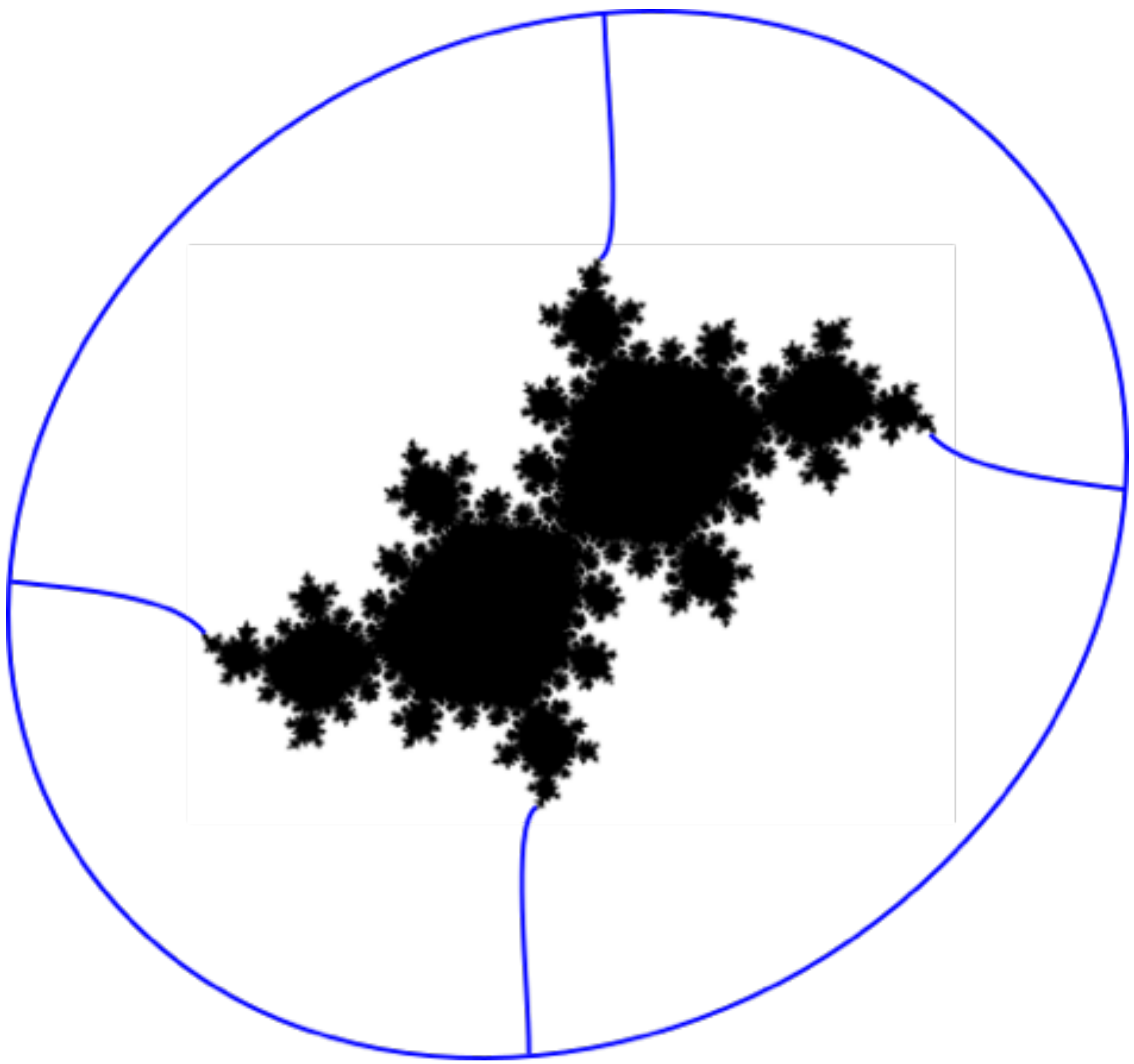

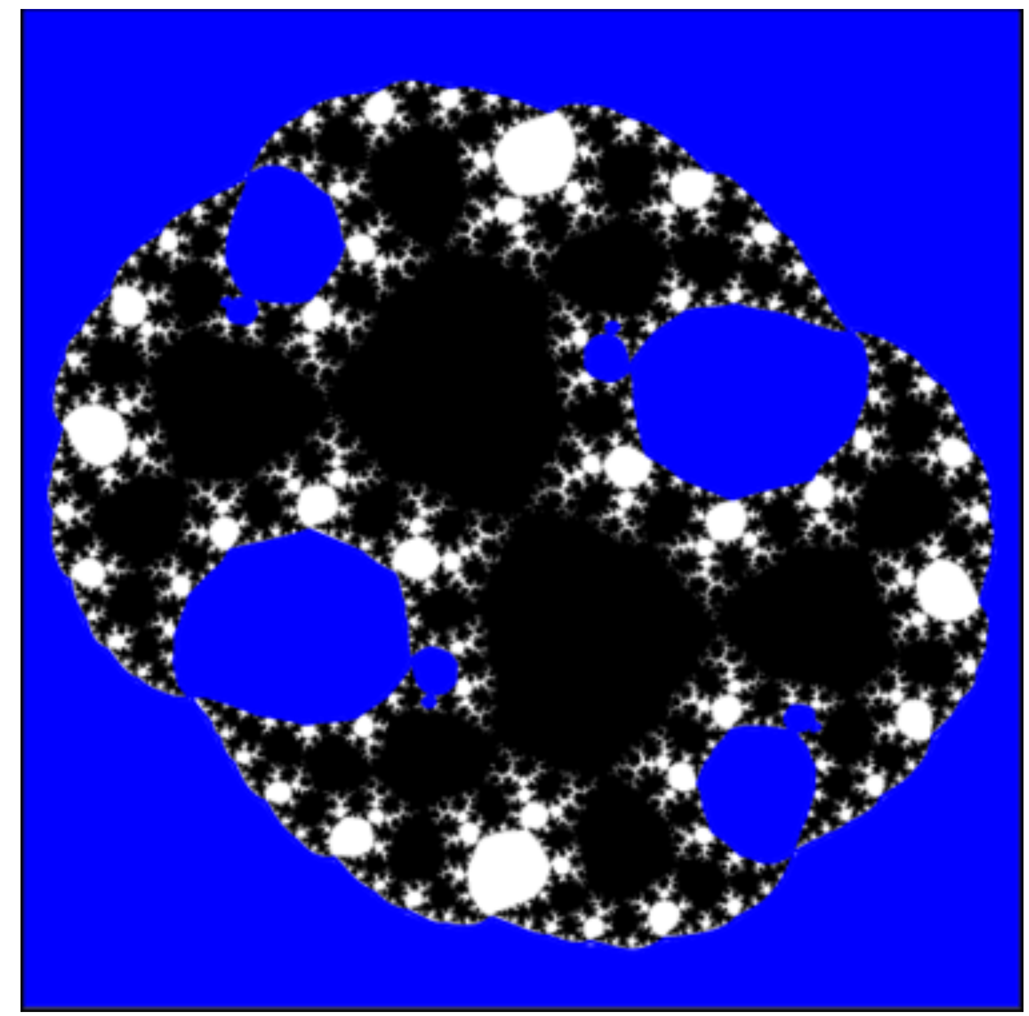

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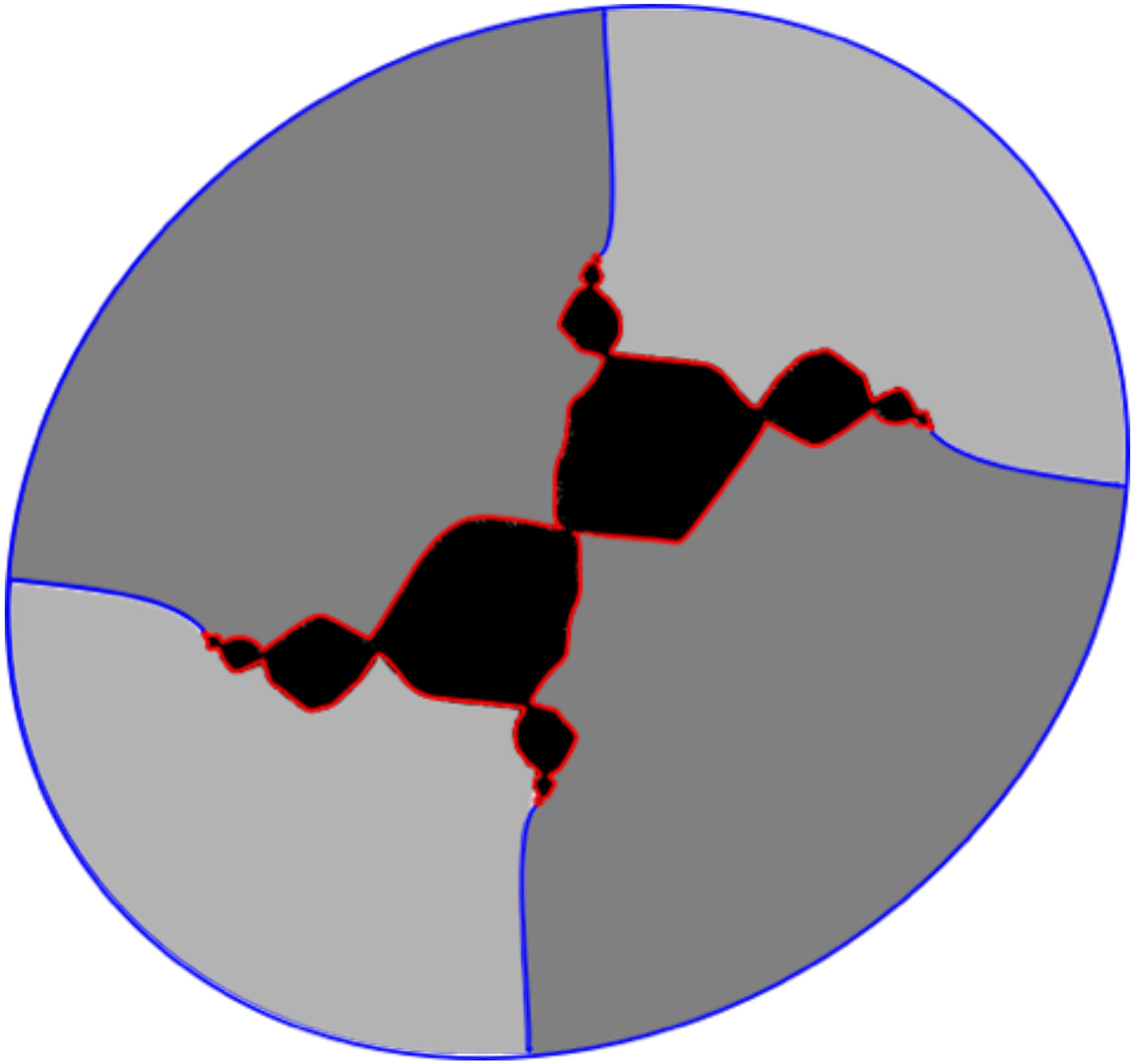

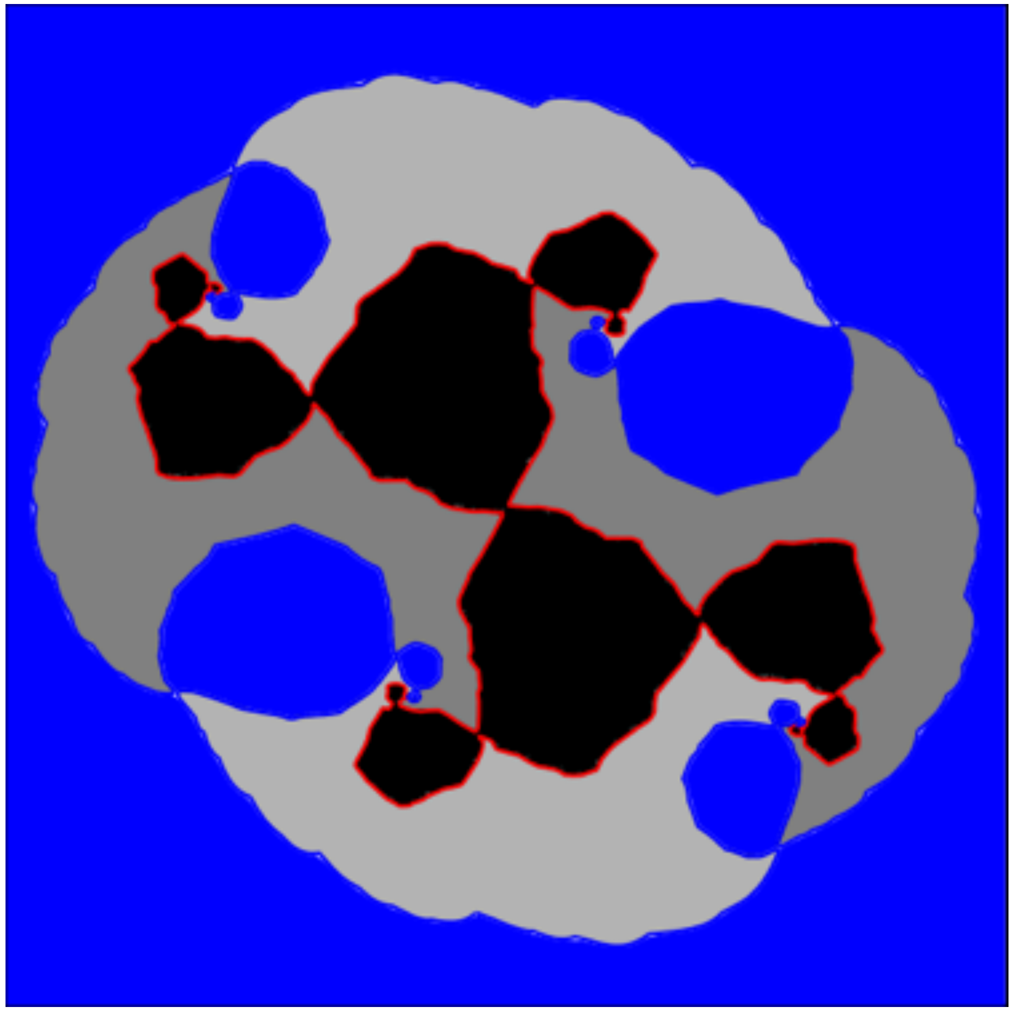

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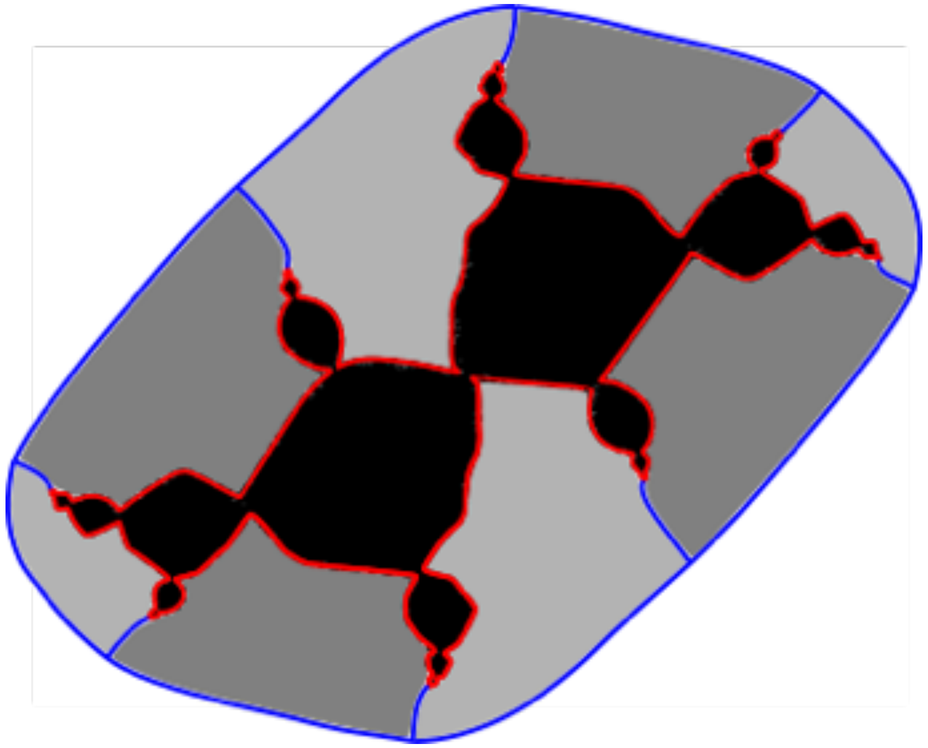

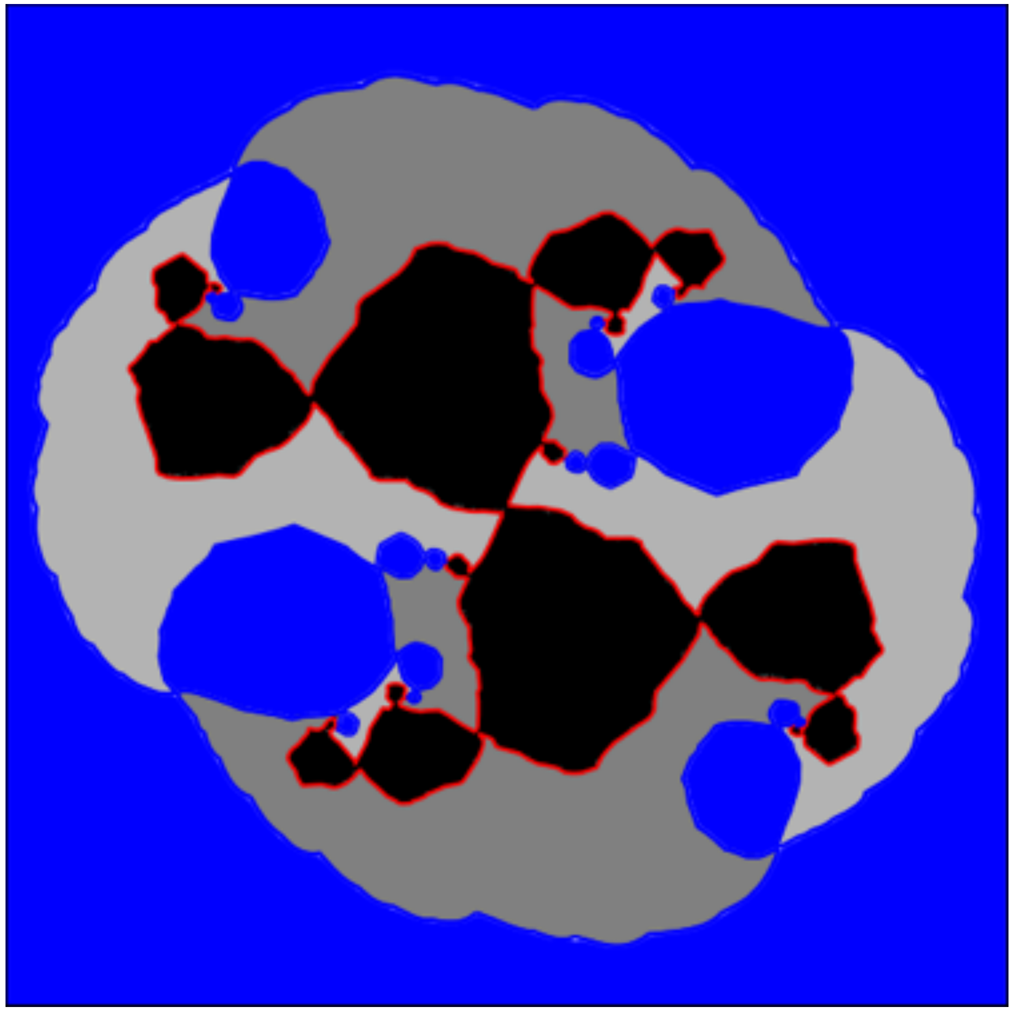

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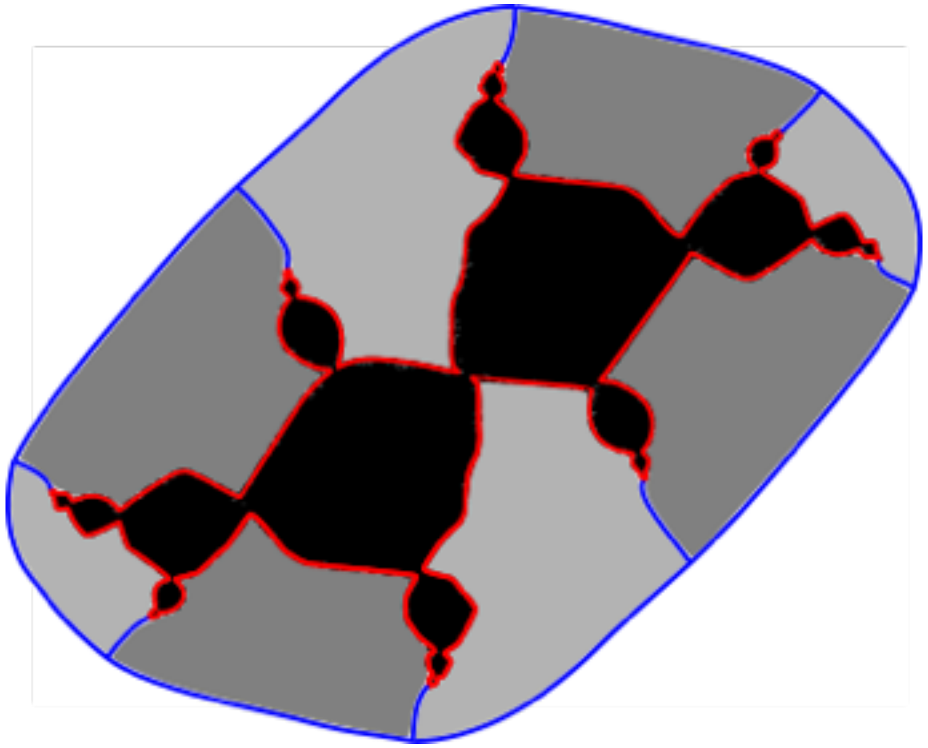

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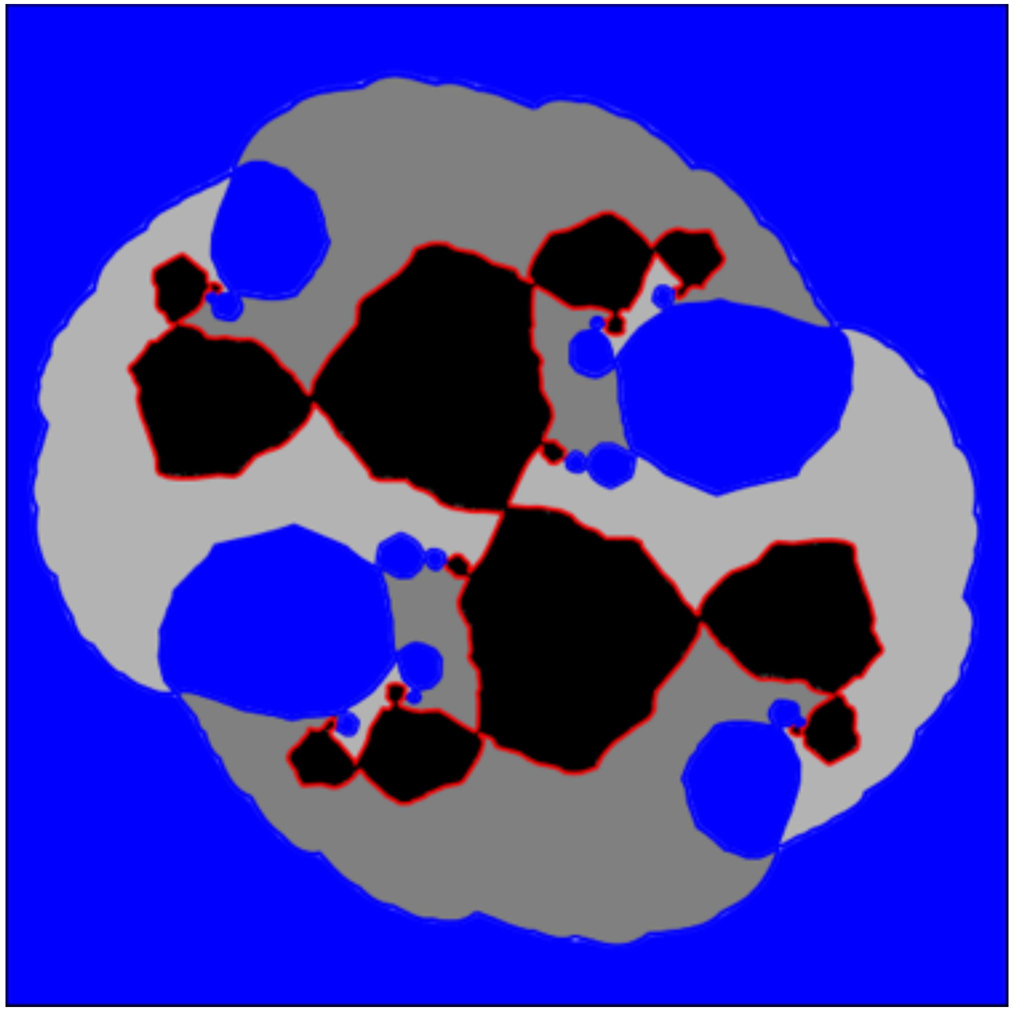

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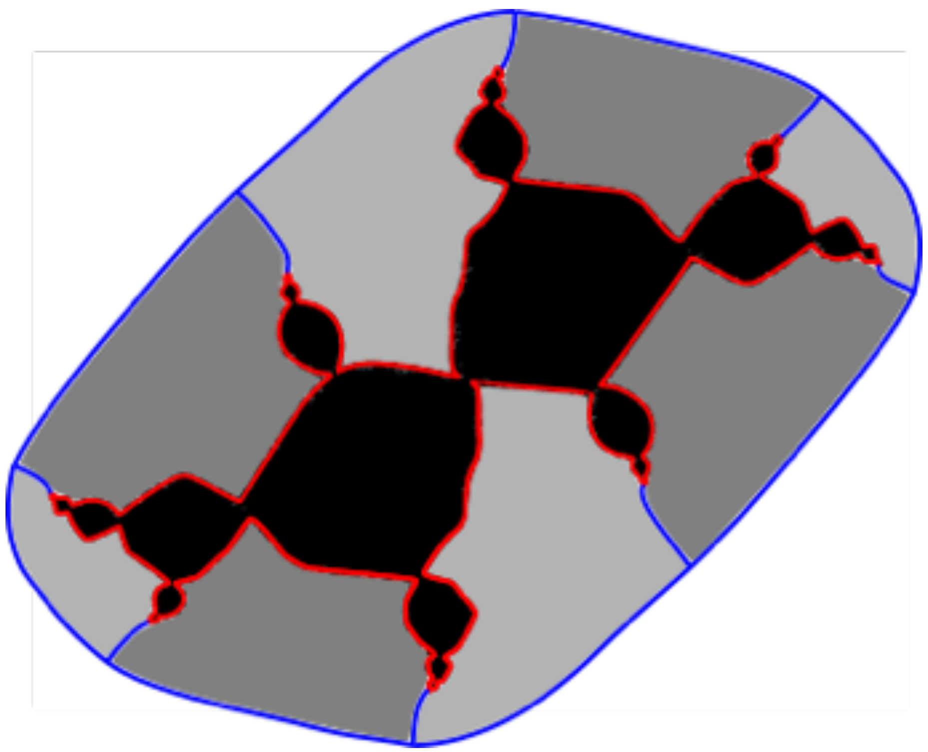

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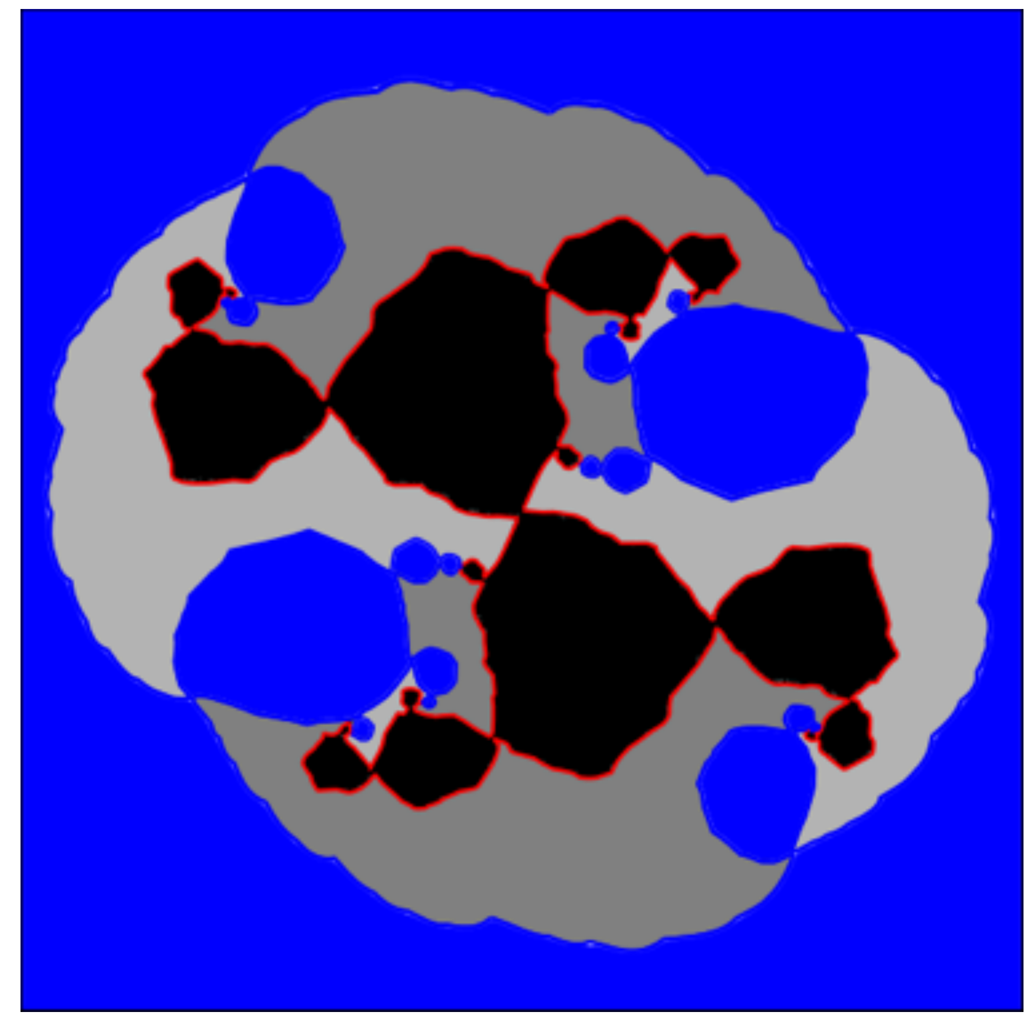

Main challenge: Prove that puzzle pieces shrink to points.

Complex A Priori Bounds [Yampolsky]

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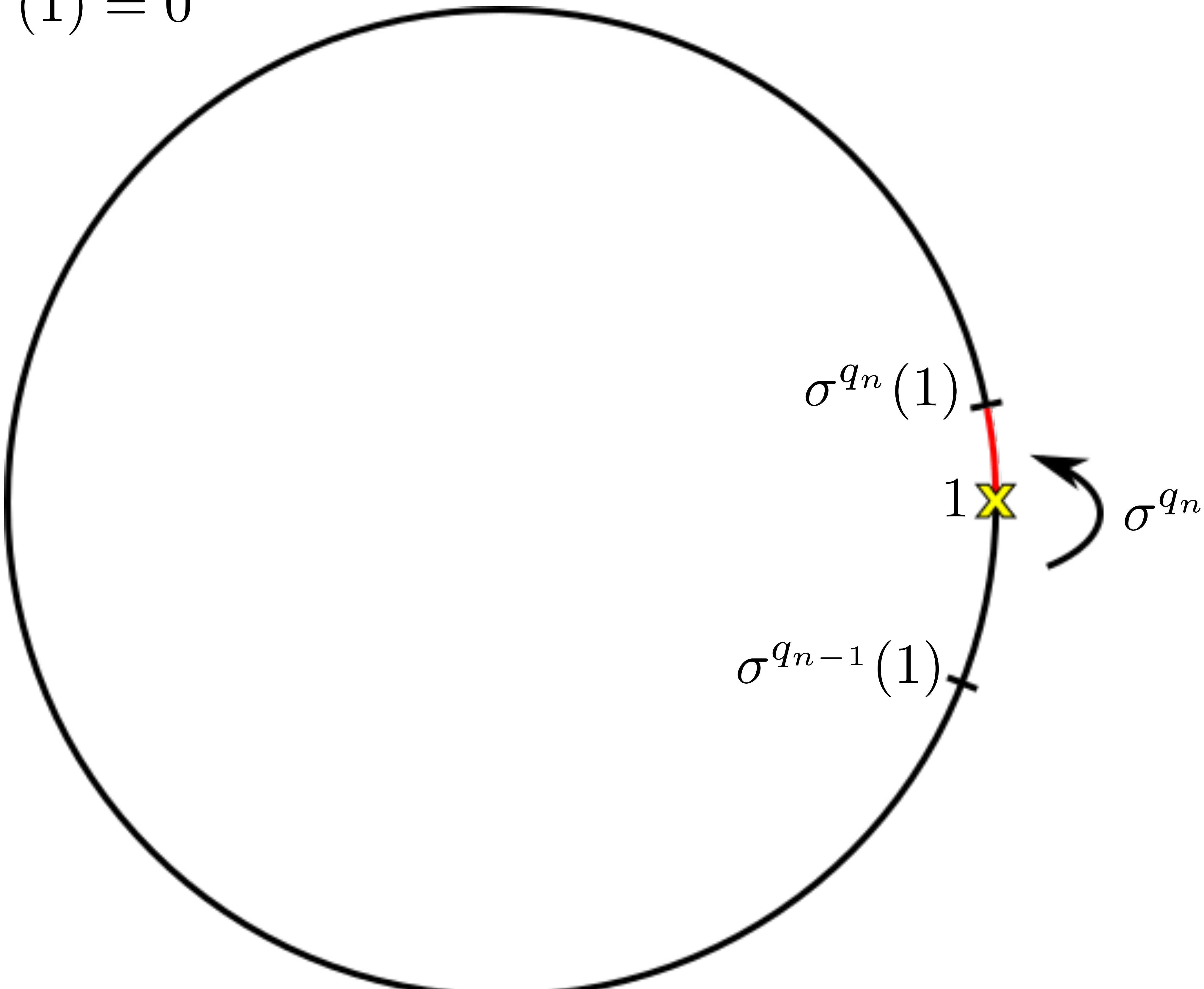
$\sigma : S^1 \to S^1$

$\sigma'(1) = \sigma''(1) = 0$

$\rho(\sigma) \notin \mathbb{Q}$

$q_1, q_2, \ldots$



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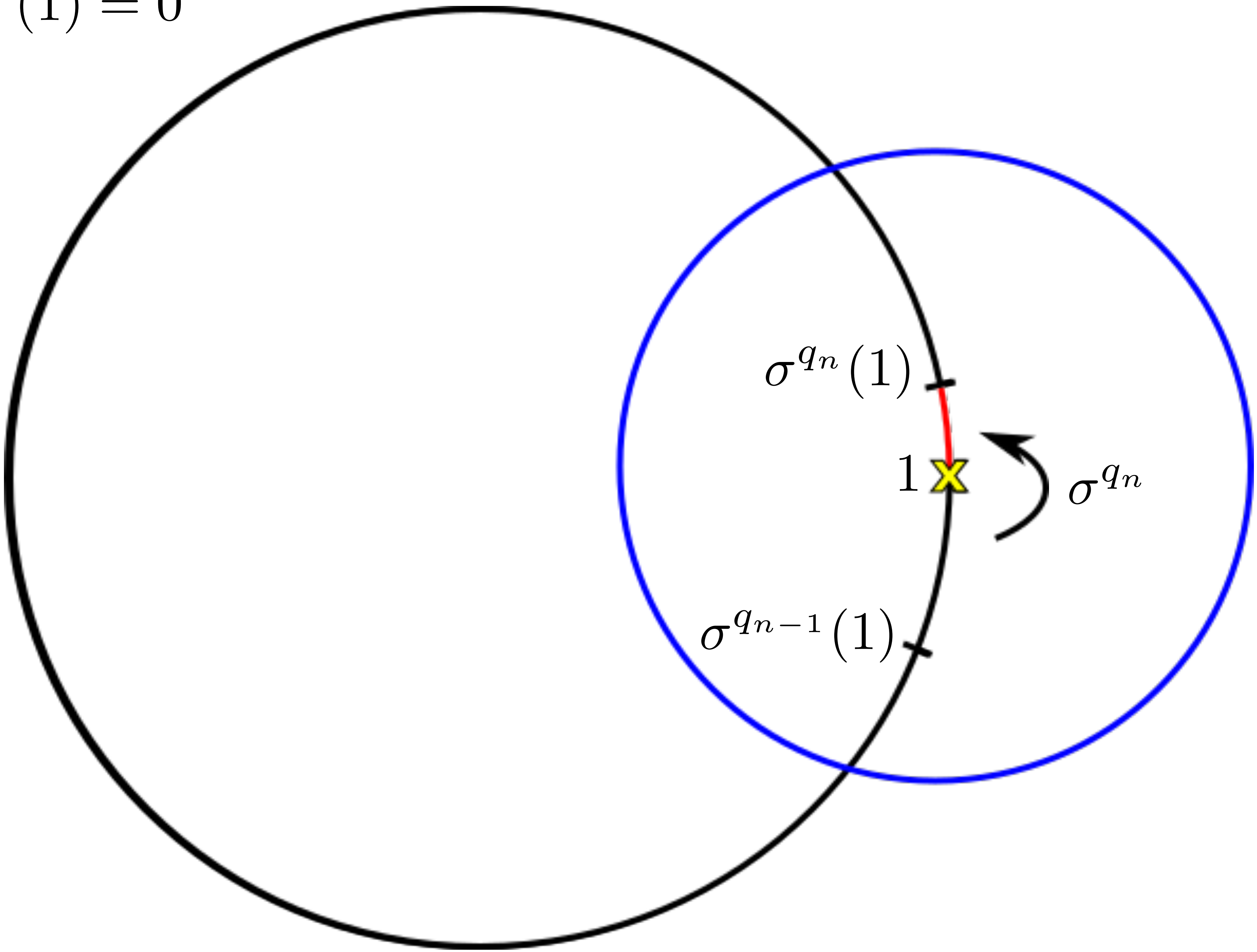

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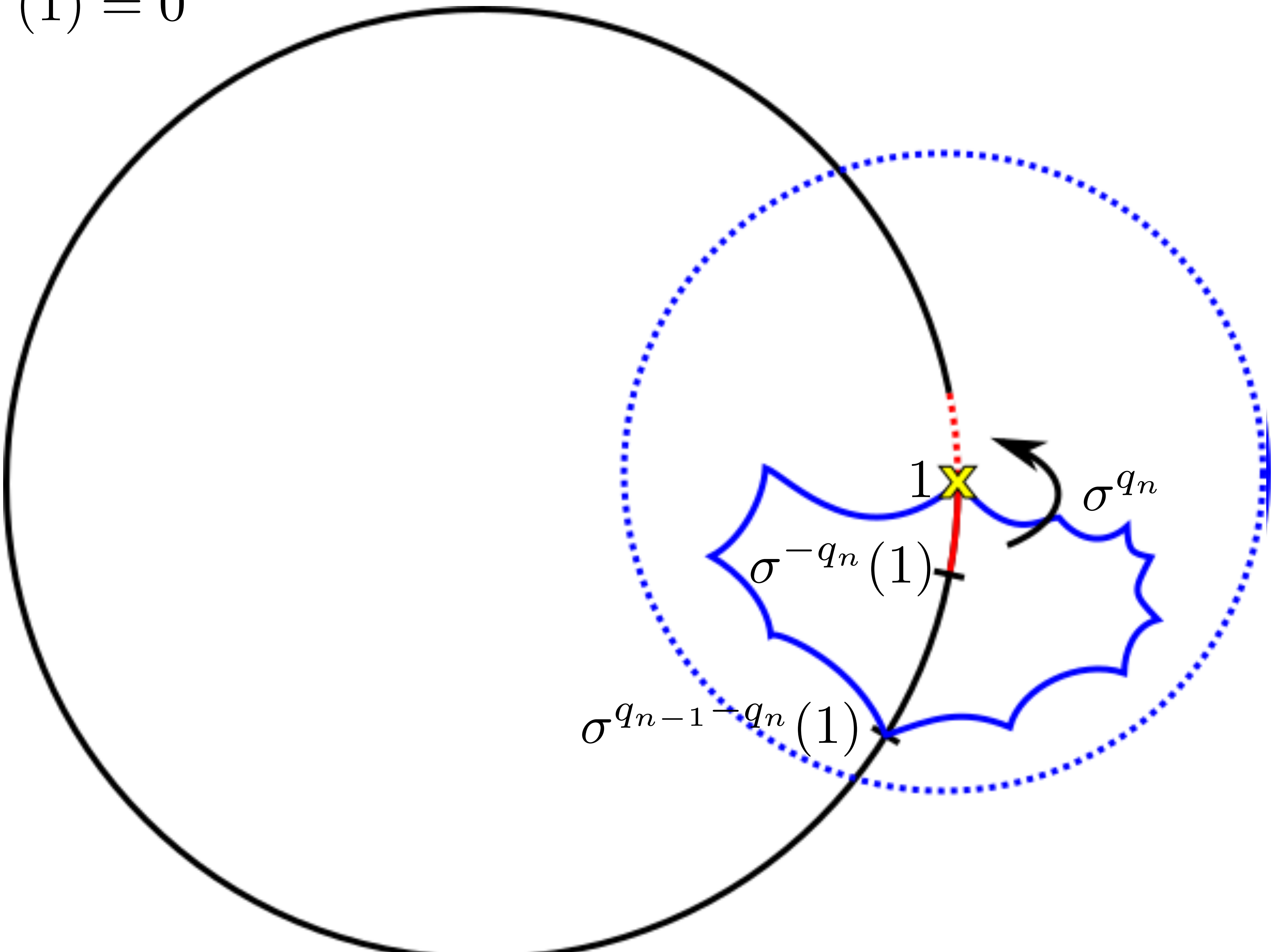
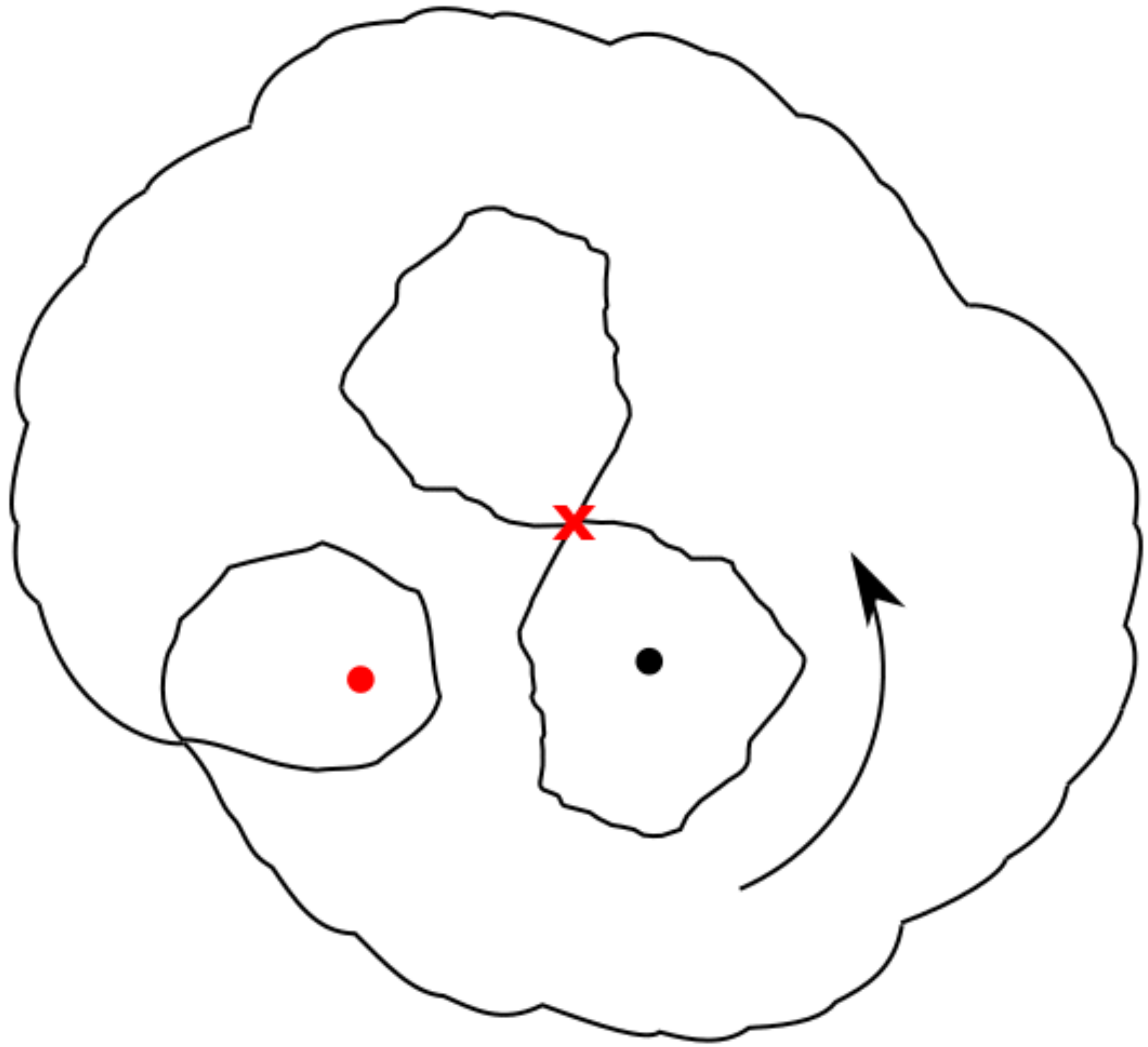


Blaschke Product Model



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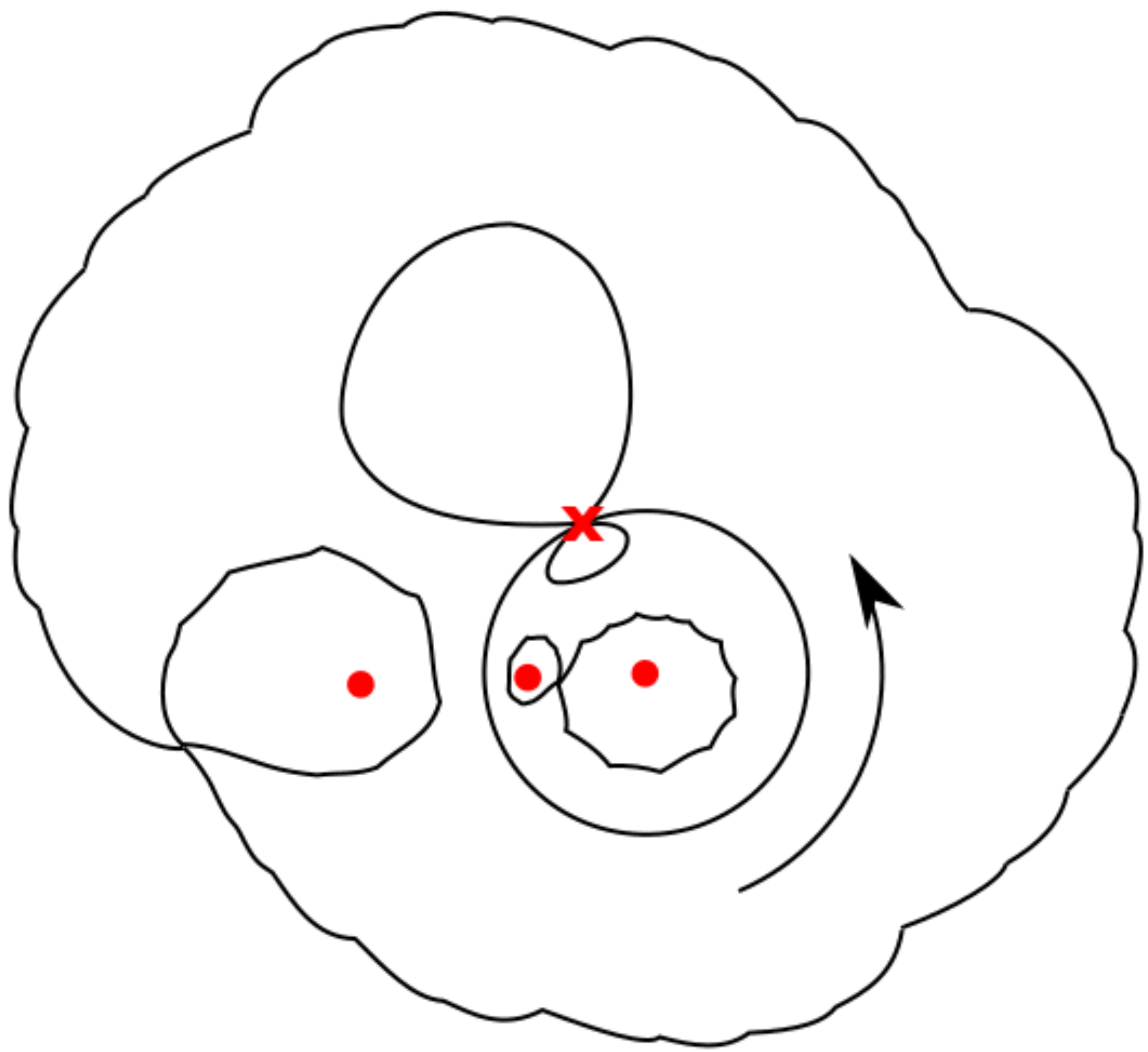
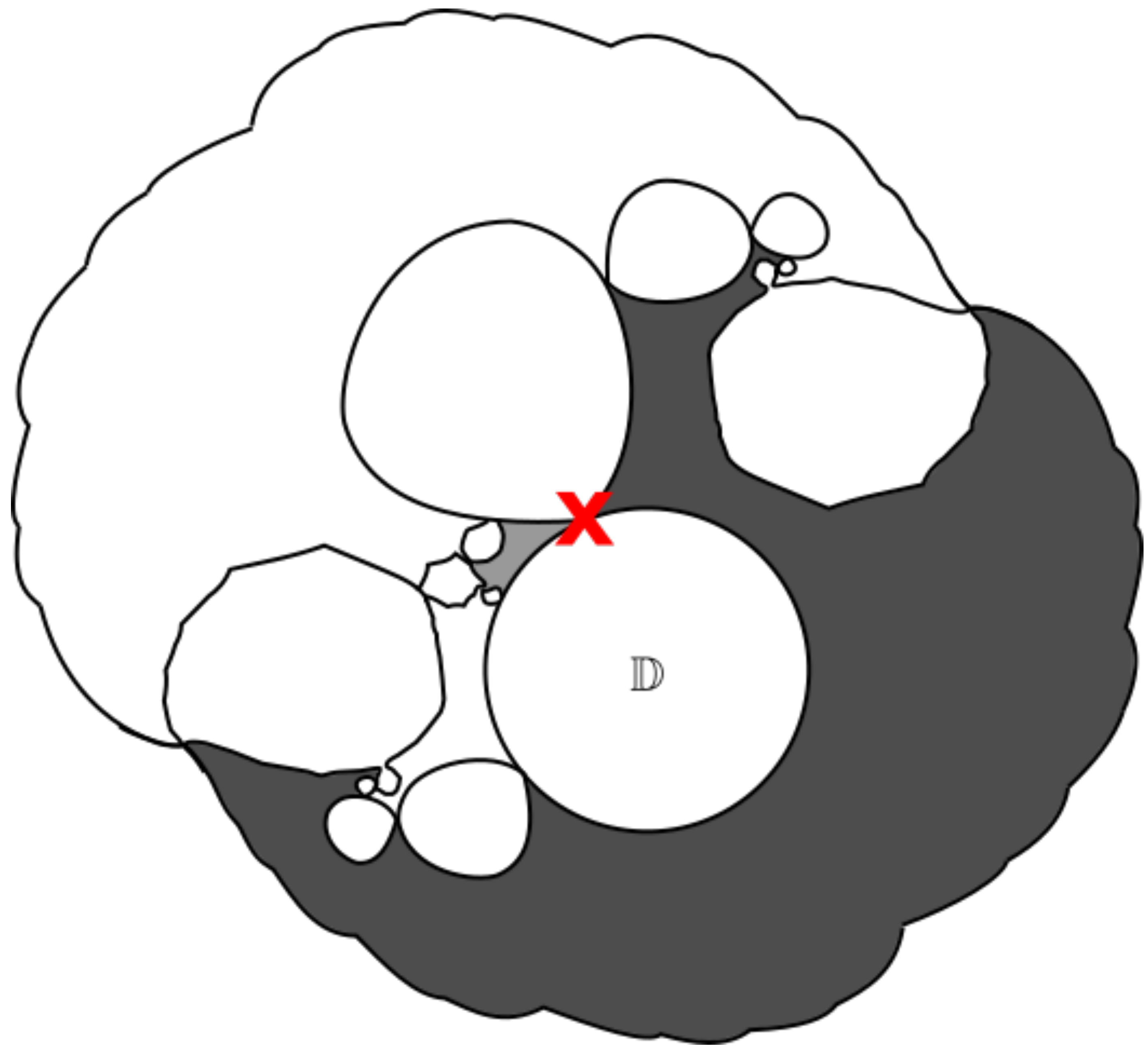


An adaptation of construction found in [Yampolsky, Zakeri].

Critical Puzzle Pieces



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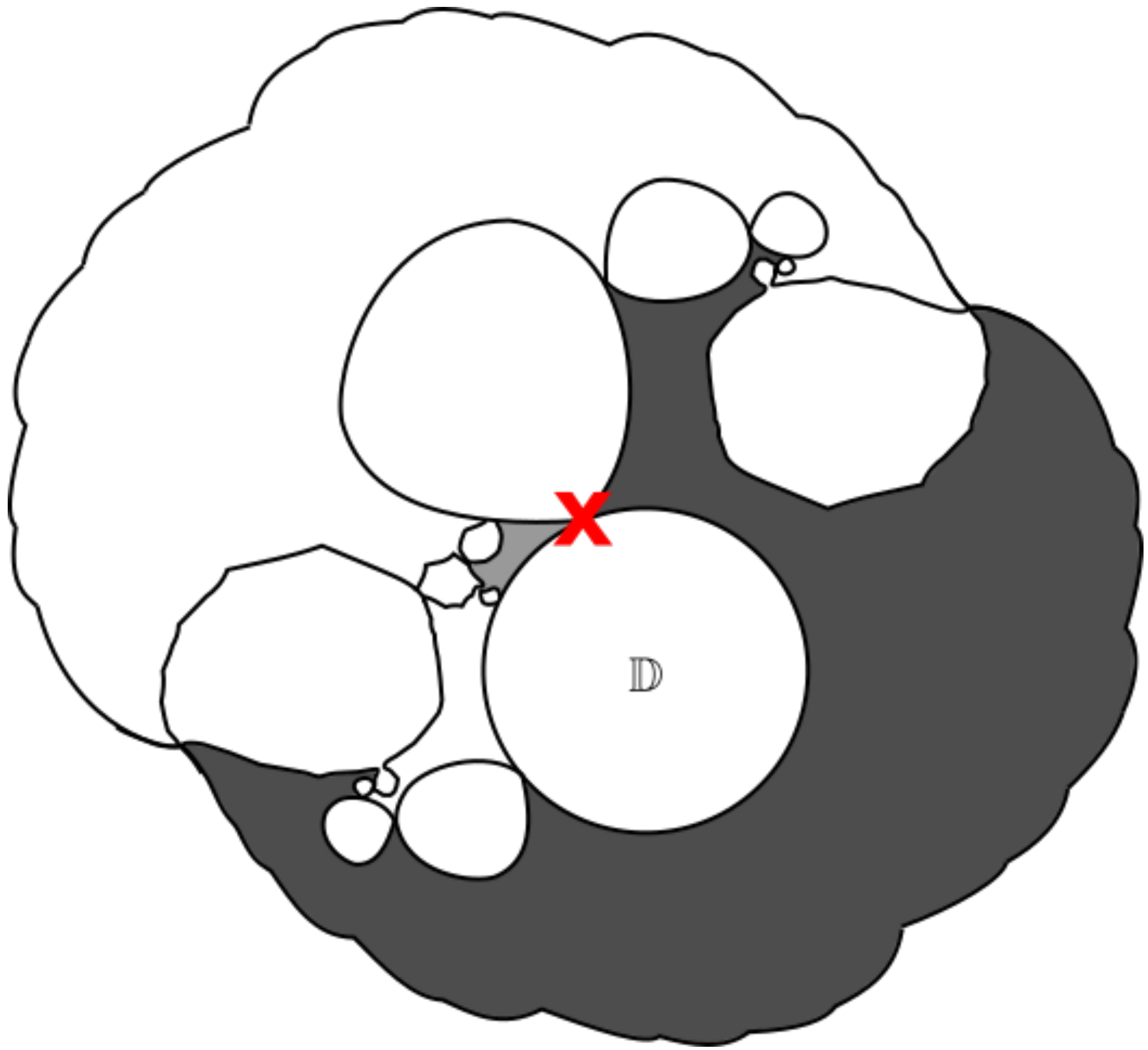

Using complex a priori bounds, can show that all puzzles shrink. Therefore, f_S and f_B are mateable.

Thank you for your attention!