An overlapping generalization of a family of self-affine carpets

Leticia Pardo Simón

Topics in Complex Dynamics
IMUB. Barcelona

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Cantor set

If $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(x) = \frac{1}{3}x; \qquad f_2(x) = \frac{1}{3}x + \frac{2}{3},$$

then $f_1(F)$ are $f_2(F)$ the left and right halves of F, so

$$F = f_1(F) \cup f_2(F).$$

Iterated function Systems

Definition

A family $\{S_1, S_2, \ldots, S_m\}$ of contractions on $\mathbb{R}^N$, i.e.

$$|S_i(x) - S_i(y)| \leq c_i|x - y| \qquad x, y \ \in \mathbb{R}^N, \ c_i < 1$$

is called an **iterated function system (IFS)**.

Theorem

Given an IFS there exists a unique, non-empty compact set F satisfying

$$F = \bigcup_{i=1}^{m} S_i(F),$$

*called the **attractor** of the IFS.*

Iterated function Systems

Moreover, for every non-empty compact set $E \in \mathcal{S}$ such that $S_i(E) \subseteq E$ for all i,

$$F = \bigcap_{k=0}^{\infty} S^k(E).$$

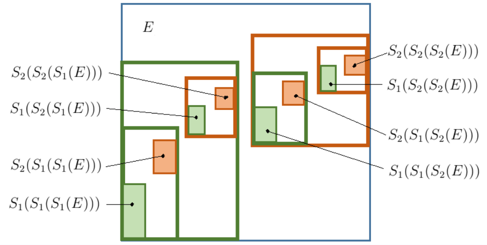


Figure: *Construction of the attractor F.The sets $S^k(E)$ give increasingly good approximations of F.*

Self-similar vs self-affine sets

If the $S_i(x) = T_i(x) + b_i$ are affine contractions, where the T_i are non-singular contracting linear mappings on $\mathbb{R}^n$ and $b_i \in \mathbb{R}^n$ are translation vectors,F is a **self-affine set**.

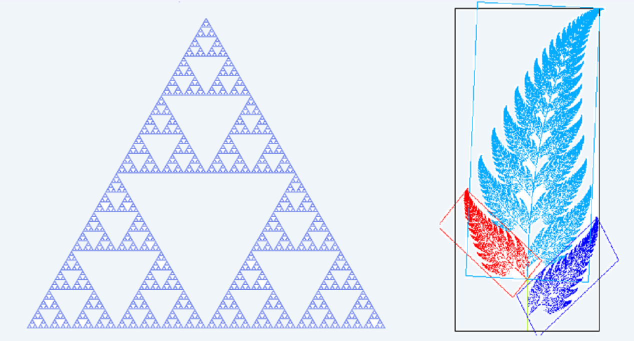
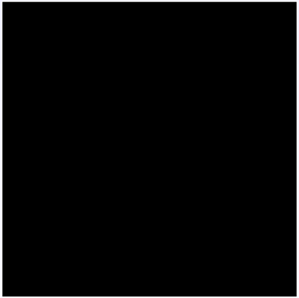


Figure: *Images from wikipedia.*

Bedford-McMullen carpet construction



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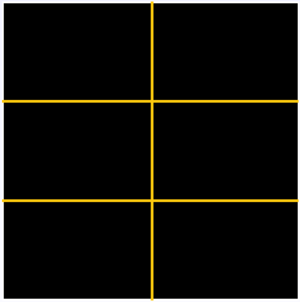

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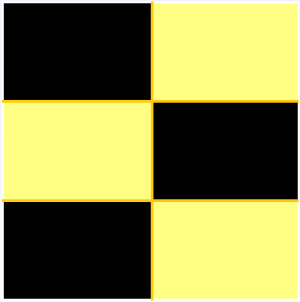

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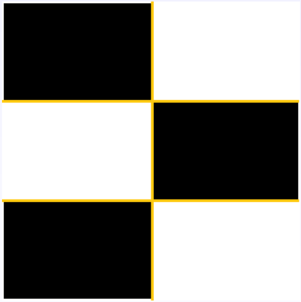

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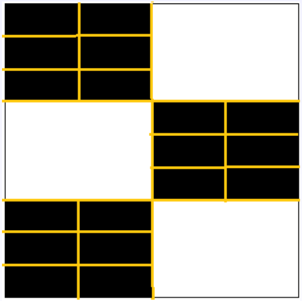
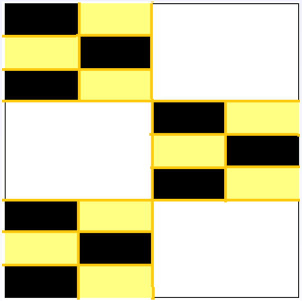


Bedford-McMullen carpet construction



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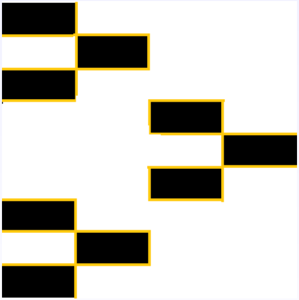
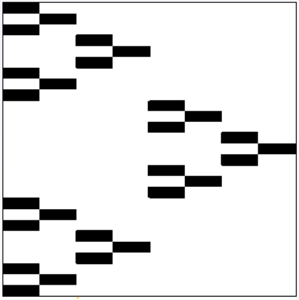
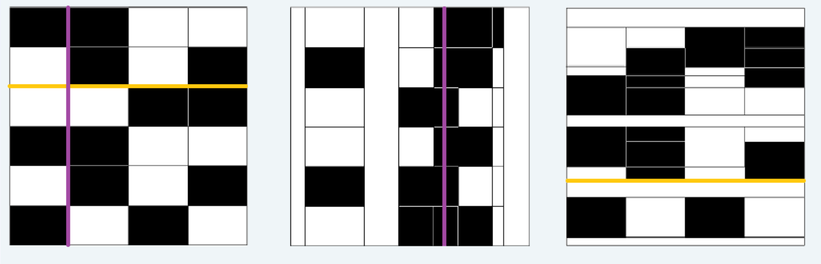


Bedford-McMullen carpet construction



Overlapping constructions

$D \subseteq D_0 = \{0, \dots, n-1\} \times \{0, \dots, m-1\}$.

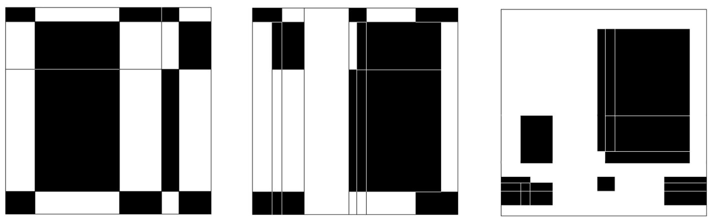


Bedford-McMullen Fraser-Shmerkin Rows overlap

$$S_{(i,j)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{i-1}{m} \\ \frac{t_j}{n} \end{pmatrix}$$

Barański overlapping carpets

$D \subseteq D_0 = \{0, \dots, n-1\} \times \{0, \dots, m-1\}.$



$$S_{\underline{t},(i,j)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ 0 & b_j \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_i \\ t_j \end{pmatrix},$$

Symbolic coding

$$
\begin{tikzcd}
(D^{\mathbb{N}}, \mu) \arrow[r, "\Pi_{\underline{t}}"] \arrow[d, "\pi_X"] & (F_{\underline{t}}, \mu \circ \Pi_{\underline{t}}^{-1}) \arrow[d, "\pi_X"] \\
(\overline{D}_X^{\mathbb{N}}, \overline{\mu}) \arrow[r, "\overline{\Pi}_{\underline{t}}"] & (\pi_X(F_{\underline{t}}), \overline{\mu} \circ \overline{\Pi}_{\underline{t}}^{-1})
\end{tikzcd}
$$

Definition

A **cylinder of level k** in $D^{\mathbb{N}}$ is a set of the form

$$[\omega_0, \ldots, \omega_k] = \{\underline{\tau} = (\tau_i)_{i=0}^{\infty} \in D^{\mathbb{N}} \text{ such that } \tau_i = \omega_i \text{ for all } 0 \leq i \leq k\}.$$

To each cylinder we can assign the **Bernoulli measure**:

$$\mu_{\underline{p}}([\omega_0, \ldots, \omega_k]) = p_{\omega_0} p_{\omega_1} \cdots p_{\omega_k}.$$

Hochman's results

Suppose the IFS $\mathcal{I} = \{S_i(x) = ax + t_i\}_{i \in A}$ does not have super-exponential concentration of cylinders. Let $p = (p_i)_{i \in A}$ be a probability vector, and let ν be the self-similar measure associated to the IFS $\mathcal{I}$ and the vector p,

$$\dim_H F = min\left(\frac{\log |A|}{\log(1/a)}, 1\right).$$

Let A be a finite index set and fix $a \in (0, 1/2)$.

- The family of $(t_i)_{i \in A}$ such that the IFS $\mathcal{I} = \{ax + t_i\}_{i \in A}$ has super-exponential concentration of cylinders has Hausdorff and packing dimension $|A| - 1$.

Preliminaries

Billingsley's Lemma

If μ is a finite measure on $\mathbb{R}^d$, $A \subseteq \mathbb{R}^d$ with $\mu(A) > 0$, and suppose that for some integer base $b \geq 2$,

$$\alpha_1 \leq \liminf_{n\to\infty} \frac{\log \mu(\mathcal{D}^d_{b^n}(x))}{-n \log b} \leq \alpha_2$$

for every $x \in A$, then $\alpha_1 \leq \dim_H(A) \leq \alpha_2$.

Result

Theorem

There exists a set $E \subseteq A$ of Hausdorff dimension $|\overline{D}_Y| + |\overline{D}_X| - 1$ (in particular of zero $|\overline{D}_X| + |\overline{D}_Y|$-dimensional Lebesgue measure) such that

$$\dim_H(F_{\underline{t}}) = \max_{\boldsymbol{p} \in \mathbb{P}^{|D|}} g(\boldsymbol{p}) \qquad \textit{if } \underline{t} \in A \setminus E$$

where

$$g(\mathbf{p}) = \begin{cases} \dfrac{\sum_{i=1}^{n} R_i(\mathbf{q}) \log R_i(\mathbf{p})}{\sum_{i=1}^{n} R_i(\mathbf{p}) \log a_i} + \dfrac{\sum_{ij} p_{ij} \log\left(\frac{p_{ij}}{R_i(\mathbf{p})}\right)}{\sum_{j=1}^{m} S_j(\mathbf{p}) \log b_j} & \text{if } \mathbf{p} \in \mathcal{S}_{\mathcal{A}} \\[2ex] \dfrac{\sum_{j=1}^{m} S_j(\mathbf{p}) \log S_j(\mathbf{p})}{\sum_{j=1}^{m} S_j(\mathbf{p}) \log b_j} + \dfrac{\sum_{ij} p_{ij} \log\left(\frac{p_{ij}}{S_j(\mathbf{p})}\right)}{\sum_{i=1}^{n} R_i(\mathbf{p}) \log a_i} & \text{if } \mathbf{p} \in {\mathcal{S}_{\mathcal{A}}}^c \end{cases}$$

Idea of the proof

- Step 1: **approximation** by a system Bedford-McMullen with **uniform fibers** (UF).
- Step 2: **projection** onto the coordinate axes.
- Step 3: approximation by a **system without overlapping**, with “enough maps”.
  -Using Hotchman + Vitali covering lemma.
- Step 4: New approx with UF and induction of a **measure**.
- Step 5: Use of approx. squares + **Billingsley**’s Lemma.


THANKS FOR YOUR ATTENTION!

Valid parameters

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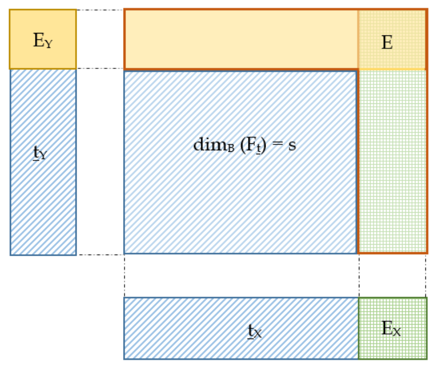

Fractal dimensions

A $\delta-$**cover** of F is a collection $\{U_i\}_{i=1}^{\infty}$ such that $F \subset \bigcup_{i=1}^{\infty} U_i$ and $diam(U_i) \leq \delta$ for each i.

Box-counting dimension

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \qquad N_\delta(F) = min\ \#\ sets\ in\ \delta-cover.$$

Hausdorff dimension

$dim_H(F) = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$,
where
$\mathcal{H}^s(F) = \lim_{\delta \to 0} \left(\inf \left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}\ is\ a\ \delta\text{-}cover\ of\ F\right\}\right)$

Packing dimension

-Analogous def to Hausdorff measure but with disjoint balls of radii δ. For self-affine sets, $\dim_B F = \dim_P F$.