Escaping Fatou components of transcendental self-maps of the punctured plane

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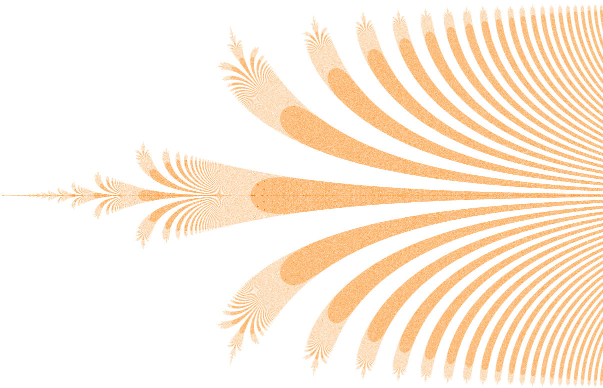
Sketch of the talk

1. Introduction to holomorphic self-maps of $\mathbb{C}^*$
2. The escaping set
3. Preliminaries on approximation theory
4. Sketch of the constructions of wandering domains and Baker domains



Transcendental self-maps of $\mathbb{C}^*$

Let $f : S \subseteq \widehat{\mathbb{C}} \to S$ be holomorphic s.t. $\widehat{\mathbb{C}} \setminus S$ are essential singularities. By Picard's theorem there are three interesting cases:

- $S = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the **Riemann sphere** (rational functions);
- $S = \mathbb{C}$, the **complex plane** (transcendental entire functions);
- $S = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the **punctured plane**.

Råd53 H. Rådström, *On the iteration of analytic functions*, Math. Scand. **1** (1953), 85–92.
Bha69 P. Bhattacharyya, *Iteration of analytic functions*, PhD Thesis (1969), University of London, 1969.

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Holomorphic self-maps of $\mathbb{C}^*$ were first studied in 1953 by Rådström.

Theorem (Bhattacharyya 1969)

Every transcendental function $f : \mathbb{C}^* \to \mathbb{C}^*$ is of the form

$$f(z) = z^n \exp\big(g(z) + h(1/z)\big)$$

for some $n \in \mathbb{Z}$ and g, h non-constant entire functions.

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The escaping set of a transcendental entire function

The **escaping set** of a transcendental entire function f,

$$I(f) := \{z \in \mathbb{C} \ : \ f^n(z) \to \infty \text{ as } n \to \infty\}$$

was introduced by Eremenko in 1989.

Theorem (Eremenko 1989, Eremenko & Lyubich 1992)

Let f be a transcendental entire function. Then,

I1. $I(f) \cap J(f) \neq \emptyset$;

I2. $J(f) = \partial I(f)$;

I3. *all the components of* $\overline{I(f)}$ *are unbounded;*

I4. *if* $f \in \mathcal{B}$, *then* $I(f) \subseteq J(f)$.

Here $\mathcal{B}$ denotes the so-called Eremenko-Lyubich class:

$$\mathcal{B} := \{f \text{ transcendental entire function } : \ \operatorname{sing}(f^{-1}) \text{ is bounded}\}.$$

Ere89 A. Eremenko, *On the iteration of entire functions*, Dynamical Systems and Ergodic Theory, Banach Center Publ. **23** (1989), 339-345.

EL92 A. Eremenko, and M. Lyubich *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), 989–1020.

The escaping set of a transcendental self-map of $\mathbb{C}^*$

If f is a transcendental self-map of $\mathbb{C}^*$, the **escaping set** of f is

$$I(f) := \{z \in \mathbb{C}^* \ : \ \omega(z,f) \subseteq \{0,\infty\}\},$$

where $\omega(z,f) := \bigcap_{n\in\mathbb{N}} \overline{\{f^k(z) : k \geqslant n\}}$. Then $I(f)$ contains

$$I_0(f) := \{z \in \mathbb{C}^* \ : \ f^n(z) \to 0 \text{ as } n \to \infty\},$$

$$I_\infty(f) := \{z \in \mathbb{C}^* \ : \ f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Mar14 D. Martí-Pete, *The escaping set of transcendental self-maps of the punctured plane*, arXiv:1412.1032, December 2014.

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We define the **essential itinerary** of a point $z \in I(f)$ to be the sequence $e = (e_n) \in \{0,\infty\}^{\mathbb{N}}$ such that

$$e_n := \begin{cases} 0, & \text{if } |f^n(z)| \leqslant 1, \\ \infty, & \text{if } |f^n(z)| > 1, \end{cases}$$

for all $n \geqslant 0$. The set of points whose essential itinerary is eventually a shift of e is

$$I_e(f) := \{z \in I(f) \ : \ \exists \ell, k \in \mathbb{N},\ \forall n \geqslant 0,\ |f^{n+\ell}(z)| > 1 \ \Leftrightarrow \ e_{n+k} = \infty\}.$$

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Eremenko's properties

Theorem (Martí-Pete 2014)

Let f be a transcendental self-map of $\mathbb{C}^*$. For each $e \in \{0, \infty\}^{\mathbb{N}}$,

- I1. $I_e(f) \cap J(f) \neq \emptyset$,
- I2. $J(f) = \partial I_e(f)$, and $J(f) = \partial I(f)$,
- I3. the connected components of $\overline{I_e(f)}$ are unbounded, and hence the connected components of $\overline{I(f)}$ are unbounded,
- I4. if $f \in \mathcal{B}^*$, then $I(f) \subseteq J(f)$.

The analog of class $\mathcal{B}$ in $\mathbb{C}^*$ is

$$\mathcal{B}^* := \{f \text{ transc. self-map of } \mathbb{C}^* \ : \ \text{sing}(f^{-1}) \text{ is bounded away from } 0, \infty\}.$$

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Escaping Fatou components

Let $U \subseteq I(f)$ be a Fatou component of a transcendental function f:

- U is a **wandering domain** of f if $f^m(U) = f^n(U)$ implies $m = n$.
- U is a **Baker domain** of period p of f if ∂U contains an essential singularity α and $f^{np}_{|U} \rightrightarrows \alpha$ as $n \to \infty$. Cowen classified them into three kinds according to whether $f^p_{|U}$ is *eventually conjugated* to
  - $z \mapsto \lambda z$, $\lambda > 1$, on $\mathbb{H}$ $\rightsquigarrow$ U is a **hyperbolic** Baker domain,
  - $z \mapsto z \pm i$, on $\mathbb{H}$ $\rightsquigarrow$ U is a **simply parabolic** Baker domain,
  - $z \mapsto z + 1$, on $\mathbb{C}$ $\rightsquigarrow$ U is a **doubly parabolic** Baker domain.

Cow81 C.C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Am. Math. Soc. **265** (1981), 69–95.

Bak87 I.N. Baker, *Wandering domains for maps of the punctured plane*, Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), no. 2, 191–198.

Kot90 J. Kotus, *The domains of normality of holomorphic self-maps of $\mathbb{C}^*$*, Ann. Acad. Sci. Fenn. Ser. A I Math. **15** (1990), no. 2, 329–340.

Muk91 A.N. Mukhamedshin, *Mapping the punctured plane with wandering domains*, Siberian Math. J. **32** (1991), no. 2, 337–339.

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Baker, Mukhamedshin and Kotus used approximation theory to construct transcendental self-maps of $\mathbb{C}^*$ with wandering domains ($e = \overline{0}$, $\overline{\infty}$, $\overline{0\infty}$) and Baker domains ($e = \overline{0}$, $\overline{\infty}$).

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Q: Are there escaping Fatou components with *any* essential itinerary?

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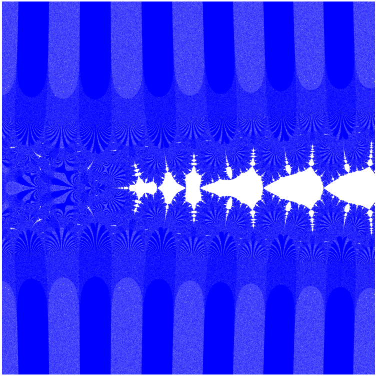
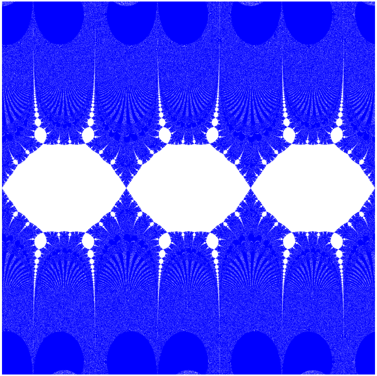
Example 1: a wandering domain

The function $f(z) = z \exp\left(\frac{\sin z}{z} + \frac{2\pi}{z}\right)$ is a transcendental self-map of $\mathbb{C}^*$ which has a bounded wandering domain escaping to ∞. Note that

$$f(z) = z + \sin z + 2\pi + o(1) \quad \text{as } \operatorname{Re} z \to \infty.$$

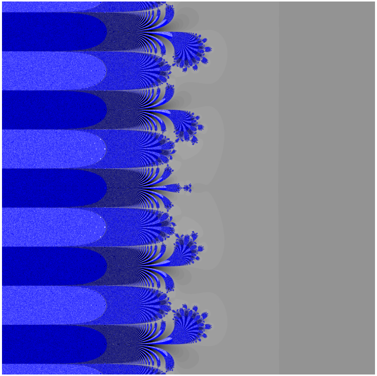


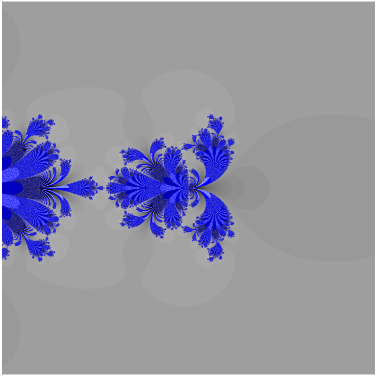


Example 2: hyperbolic Baker domains

For every $\lambda > 1$, the function $f_\lambda(z) = \lambda z \exp(e^{-z} + 1/z)$ is a transcendental self-map of $\mathbb{C}^*$ which has a hyperbolic Baker domain escaping to ∞. Note that

$$f(z) \sim \lambda z \qquad \text{as } \operatorname{Re} z \to \infty.$$

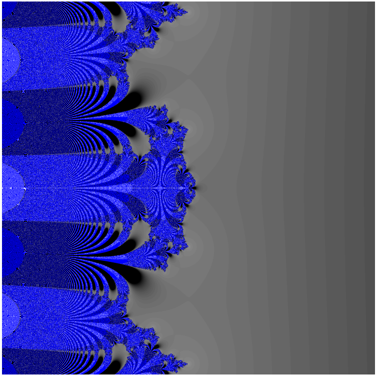


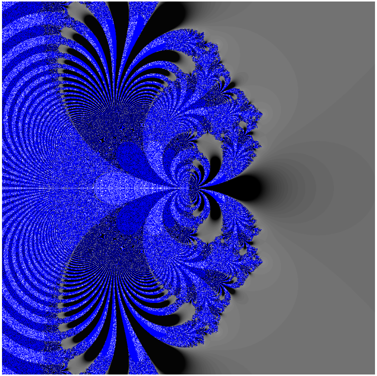


Example 3: a doubly parabolic Baker domain

The function $f(z) = z\exp\left((e^{-z}+1)/z\right)$ is a transcendental self-map of $\mathbb{C}^*$ which has a simply parabolic Baker domain escaping to ∞. Note that

$$f(z) = z + 1 + o(1) \quad \text{as } \operatorname{Re} z \to \infty.$$





Results

Theorem (Martí-Pete)

For each $e \in \{0, \infty\}^{\mathbb{N}}$ and $n \in \mathbb{Z}$, there exists a transcendental self-map of $\mathbb{C}^*$, f, such that $\mathrm{ind}(f) = n$ and $I_e(f)$ contains a wandering domain.

Theorem (Martí-Pete)

For each periodic $e \in \{0, \infty\}^{\mathbb{N}}$ and $n \in \mathbb{Z}$, there exists a transcendental self-map of $\mathbb{C}^*$, f, such that $\mathrm{ind}(f) = n$ and $I_e(f)$ contains a Baker domain, which can be taken to be hyperbolic, simply parabolic or doubly parabolic.

Remark: We can also construct *entire functions* with no zeros having escaping Fatou components (to ∞) of each of these kinds.

Mar D. Martí Pete, *Escaping Fatou components of transcendental self-maps of the punctured plane*, in preparation.

Preliminaries on approximation theory

For $0 < \alpha \leqslant 2\pi$, define the *sector* of angle α

$$W_\alpha := \{z \in \mathbb{C} \ : \ |\arg z| \leqslant \alpha/2\}.$$

Theorem (Arakeljan 1964)

Suppose $F \subseteq \mathbb{C}$ is a closed set such that $\widehat{\mathbb{C}} \setminus F$ is connected and locally connected at ∞ and F lies in a sector W_α for some $0 < \alpha \leqslant 2\pi$. Suppose $\varepsilon(t)$ is a real function, continuous and positive for $t \geqslant 0$, such that

$$\int_1^{+\infty} t^{-(\pi/\alpha)-1} \log \varepsilon(t) dt > -\infty.$$

Then every function $f : F \to \mathbb{C}$ holomorphic on $\operatorname{int} F$ and continous on F, there is an entire function g such that

$$|f(z) - g(z)| < \varepsilon(|z|), \quad \textit{for all } z \in F.$$

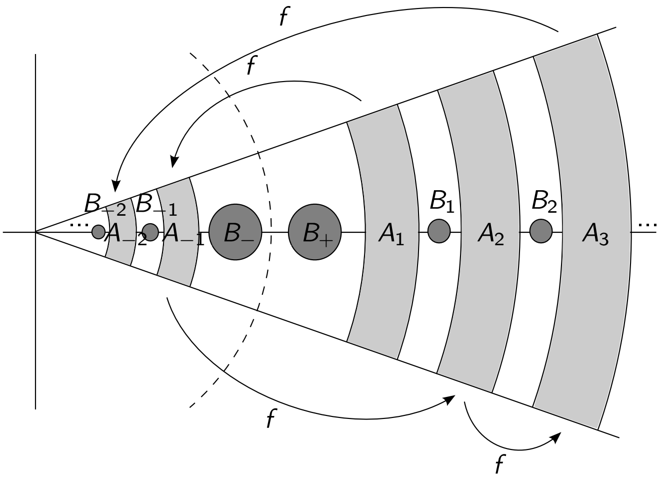
Hence, if $F \subseteq W_\alpha$ with $\alpha < \pi$ we can take $\varepsilon(|z|) = O(e^{-\gamma|z|})$ with $\gamma > 0$.

Ara64 N. U. Arakeljan, *Uniform and asymptotic approximation by entire functions on unbounded closed sets* (Russian), Dokl. Akad. Nauk SSSR **157** (1964), 9–11.

Gai87 D. Gaier, *Lectures on complex approximation*, translated from the German by Renate McLaughlin. Birkhäuser Boston, Inc., Boston, MA, 1987.

Sketch of the wandering domain construction

We use Arakeljan's theorem to construct two entire functions g and h such that $f(z) = z^n \exp\big(g(z) + h(1/z)\big)$ has a wandering domain in $I_e(f)$ which lies in a sector W_α with $0 < \alpha < \pi/2$.



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We first construct an *entire* function $f(z) = \exp\big(g(z)\big)$ such that

$$f(z) \sim \lambda z \quad \text{as} \quad \operatorname{Re} z \to +\infty,$$

for some $\lambda > 1$, and $f(\mathbb{H}_r) \subseteq \mathbb{H}_r$ for sufficiently large r. We need

$$|g(z) - \log(\lambda z)| < \varepsilon(|z|) \quad \text{on} \quad \mathbb{H}_r = \{z \in \mathbb{C} \ : \ \operatorname{Re} z > r\}.$$

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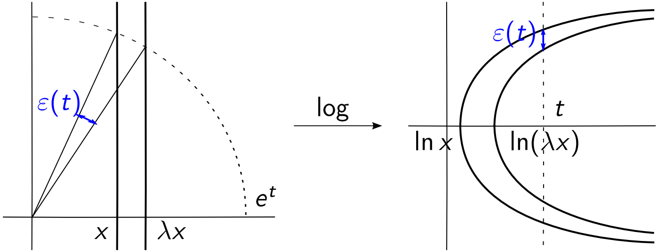

The largest error function that we can have is $\varepsilon(t) = \sqrt{2}e^{-t}$.

Sketch of the Baker domain construction II

Arakelian's theorem tells us that we can only construct g with error $\varepsilon(t)$ in a sector of angle $0 < \alpha < \pi$. Thus, we semiconjugate λz with $\sqrt{z}$:

$$\begin{array}{ccc} \mathbb{H}_r & \xrightarrow{\lambda z} & \mathbb{H}_{\lambda r} \\ \big\downarrow \sqrt{z} & & \big\downarrow \sqrt{z} \\ \sqrt{\mathbb{H}_r} & \xrightarrow{\sqrt{\lambda} z} & \sqrt{\mathbb{H}_{\lambda r}} \end{array}$$

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\end{tikzcd}$$

The new error function is

$$\tilde{\varepsilon}(t) = \frac{\varepsilon(2t)}{2} = 2\sqrt{2}e^{-2t}.$$

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Therefore we can construct an entire function g such that

$$|g(z) - \log(\sqrt{\lambda z^2})| < \varepsilon(|z|) \quad \text{on} \quad \sqrt{\mathbb{H}_r} = \{z \in \mathbb{C} \ : \ \operatorname{Re} z^2 > r\}.$$

Then $f(z) = \exp(g(z))$ satisfies that $f(\sqrt{\mathbb{H}_r}) \subseteq \sqrt{\mathbb{H}_r}$ for sufficiently large r, and hence f has a fixed hyperbolic Baker domain containing $\sqrt{\mathbb{H}_r}$ (that escapes to ∞).

Sketch of the Baker domain construction III

In order to construct a transcendental self-map of $\mathbb{C}^*$, let $\phi(z) = e^{-2z}$, and use Arakeljan's theorem on tangential approximation to construct an entire function g such that

$$\begin{cases} \left| g(z) - \frac{\log(\lambda z) - n \log z}{\phi(1/z)} \right| < \sqrt{2} e^{-4|z|}, & \text{on } \sqrt{\mathbb{H}_r}, \\ |g(z)| < 1, & \text{on } \mathbb{D}. \end{cases}$$

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Then the function

$$f(z) = z^n \exp\Big(g(z)\phi(1/z) + g(1/z)\phi(z) \Big)$$

is a transcendental self-map of $\mathbb{C}^*$ with two fixed hyperbolic Baker domains, one escaping to 0 and the other one to ∞.

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is a transcendental self-map of $\mathbb{C}^*$ with two fixed hyperbolic Baker domains, one escaping to 0 and the other one to ∞.

Finally, if you want to have a 2-periodic Baker domain U with $e = \overline{\infty 0}$, define

$$\tilde{f}(z) = 1/f(z)$$

so that $U \rightleftarrows 1/U$ by $\tilde{f}$.

THE END

Thank you for your attention!