Constructing Entire Functions (a summary)

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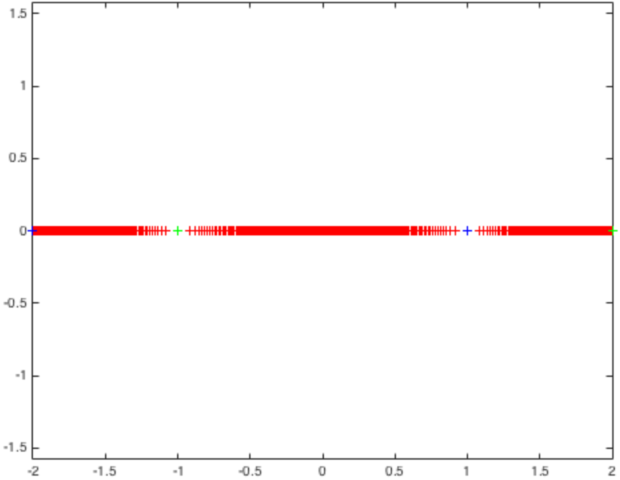

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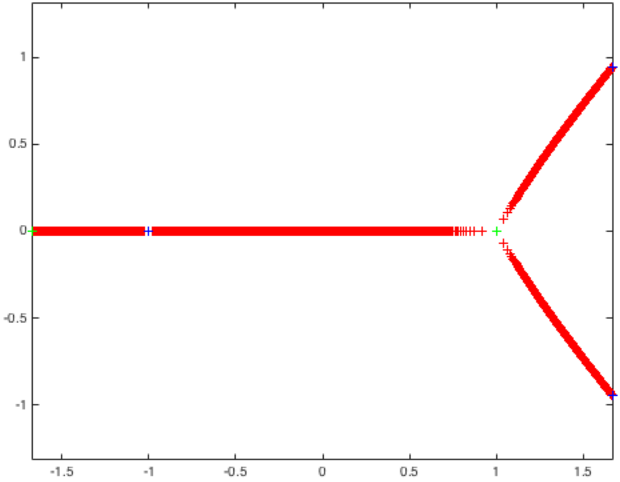

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Theorem *(Bishop)***:** Any continua can be ϵ-approximated in the Hausdorff metric by some $p^{-1}[-1,1]$.

trees $\Longleftrightarrow$ *Shabat* polynomials

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$\mathcal{S}_{2,0}$ - transcendental functions with two critical values ± 1 and no asymptotic values

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critical values: ± 1

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A Fatou component U is called *wandering* if $f^n(U) \cap f^m(U) = \emptyset$ for all $n, m \in \mathbb{N}, n \neq m$

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Theorem: *(Golberg and Keen, Eremenko and Lyubich) Functions in the Speiser Class don't have wandering domains.*

The Eremenko-Lyubich class $\mathcal{B}$ consists of those transcendental functions with bounded singular set.

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$z_n = a_n + i\pi$, $D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
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(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
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Theorem:

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Theorem: *For every choice of parameters* $\lambda, (d_n), (w_n)$ *with* $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$

$S^+ = \{x+iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1,1]$ by cosh.

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Theorem: *For every choice of parameters $\lambda, (d_n), (w_n)$ with $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$, there exists a transcendental f and a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ so that:*

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$$f(z) = \begin{cases} \cosh(\lambda \sinh(\phi(z))) & \text{if } \phi(z) \in S^+ \\ \rho_n((\phi(z) - z_n)^{d_n}) & \text{if } \phi(z) \in D_n \end{cases}$$

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(4) $\phi(0) = 0, \phi(\mathbb{R}) = \mathbb{R}$ and ϕ is conformal in S^+.

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