Fatou's web and non-escaping endpoints

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Fatou's function

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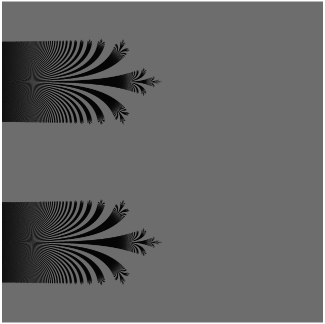

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The **fast escaping set**, $A(f) \subset I(f)$, consists of the points that go to infinity as quickly as possible under iteration.

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- $A(f)$ consists of the curves in $J(f)$ except for some of their endpoints.

Spiders' webs

Rippon and Stallard showed that for many transcendental entire functions the escaping set has a structure called a **spider's web**.

Definition 1

A set E is an (infinite) spider's web if:

1) E is connected and
2) $\exists$ a sequence $(G_n), n \in \mathbb{N}$, of bounded, simply connected domains such that
    - $G_n \subset G_{n+1}, n \in \mathbb{N}$,
    - $\partial G_n \subset E, n \in \mathbb{N}$,
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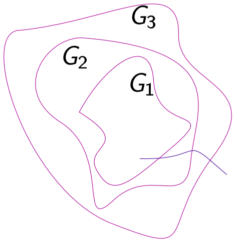

Spiders' webs

- Rippon and Stallard showed that when $I(f)$ **contains** a SW then it **is** a SW.
- In most examples we show that $A(f)$ is a SW which implies that $I(f)$ is a SW.
- There exists a complicated example of a function for which $I(f)$ is a SW whereas $A(f)$ is not, due to Rippon and Stallard.

Fatou's web

Theorem 1

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Let f be a t.e.f. and (a_n) be a positive sequence such that:

(1) $a_n \to \infty$ as $n \to \infty$,

(2) the disc $D(0, a_n)$ contains a periodic cycle of f, for all $n \in \mathbb{N}$.

Consider the set

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Theorem 2

Let f be a t.e.f. If (a_n) satisfies (1), (2) and $I(f, (a_n))^c$ has a bounded component, then $I(f)$ is a SW.

Fatou's web

Now we apply Theorem 2. Take

$$a_n = \frac{n+6}{2}, n \in \mathbb{N}.$$

Then

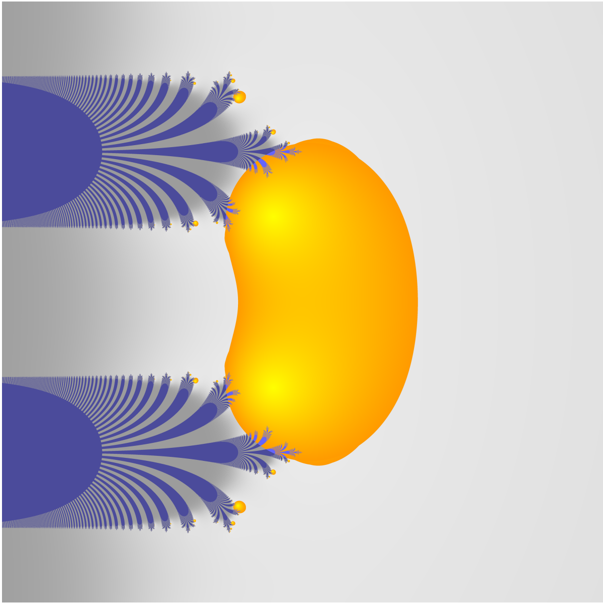
(1) $(n+6)/2 \to \infty$ as $n \to \infty$,

(2) $D(0,((n+6)/2)) \supset D(0,7/2) \supset \pm\pi i, n \in \mathbb{N}$, and

(3) All the components of $I(f,((n+6)/2))^c$ are bounded.

Hence Theorem 2 $\Rightarrow$ Theorem 1. ∎

Fatou's web



Generalisation

A similar argument can show that the function $f(z) = 2z + 2 - \log 2 - e^z$ that was first studied by Bergweiler has the same property, that is, $I(f)$ is a spider's web.

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We deduce that the same result holds for functions of the form

$$z \mapsto az + b + ce^{dz},$$

where $a, b, c, d \in \mathbb{R}$ satisfy some specific properties.

Endpoints

In 1988 Mayer showed that for the exponential family $f_a(z) = e^z + a, a < -1$, the set of all endpoints of $J(f_a)$ is totally disconnected whereas the union of the endpoints with ∞ is a connected set.
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The Julia set for Fatou's function is also a Cantor bouquet and hence we can consider the set of endpoints of $J(f)$, which we denote by $E(f)$. Mayer's result holds also for Fatou's function.

Theorem 3

Let $f(z) = z + 1 + e^{-z}$. Then $E(f)$ is totally disconnected but $E(f) \cup \{\infty\}$ is connected.

The proof is based on a result of Barański.

Endpoints

The fact that $I(f)$ is a SW for Fatou's function leads to a result about the non-escaping endpoints of $J(f)$, $\hat{E}(f) = E(f) \setminus I(f)$.

Theorem 4

Let $f(z) = z + 1 + e^{-z}$. Then $\hat{E}(f) \cup \{\infty\}$ is totally disconnected.

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Let $f(z) = z + 1 + e^{-z}$. Then $\hat{E}(f) \cup \{\infty\}$ is totally disconnected.

Proof.

Suppose there is a non-trivial component of $\hat{E}(f) \cup \{\infty\}$. Since $I(f)$ is a SW, any non-escaping endpoint is separated from ∞ by a 'loop' in $I(f)$ and so this component must lie in $\hat{E}(f) \subset E(f)$. Since, by Theorem 3, $E(f)$ is totally disconnected,we obtain a contradiction. ■

THANK YOU!