On the escaping set of exponential maps

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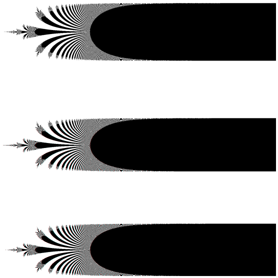

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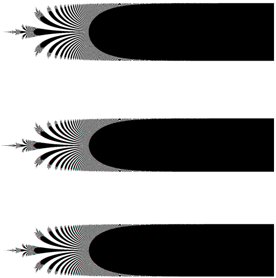

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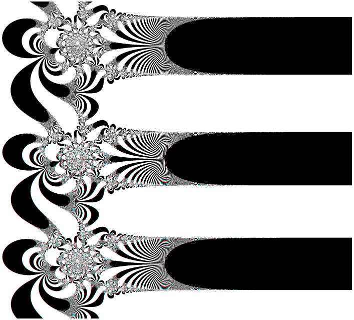

1.2 The case $a = -0.99 + 0.0001i$

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Lasse Rempe-Gillen has given an answer to this question:

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- $L_{a,\sigma} \colon \mathbb{H}_\sigma \to S_\sigma$ the branch of f_a^{-1} in S_σ, where $\sigma \in \{+, -\}$

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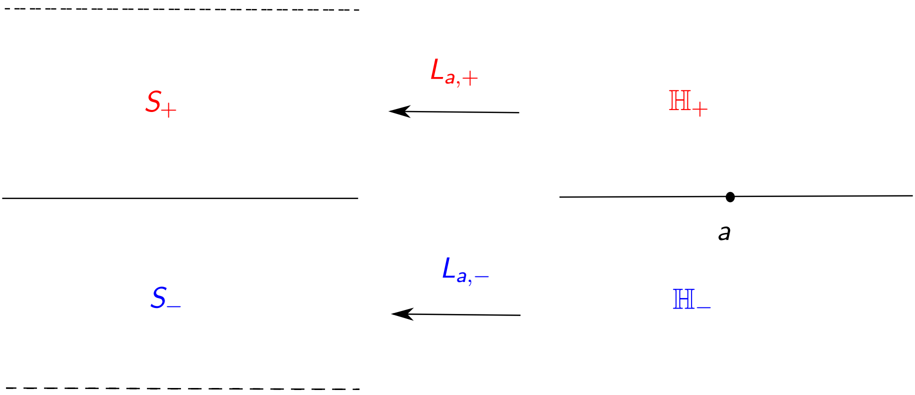

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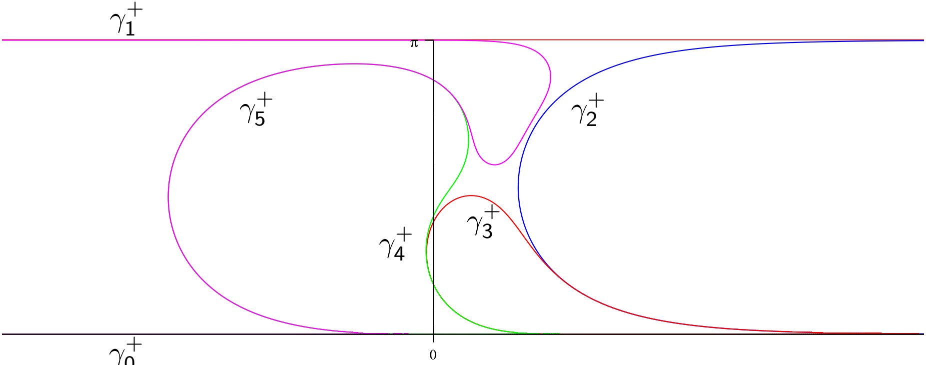
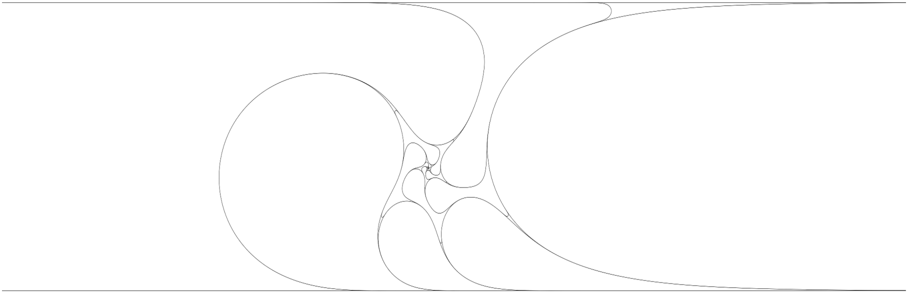


Construction of $\overline{\Gamma^+}$.

By continuing this procedure, we obtain:



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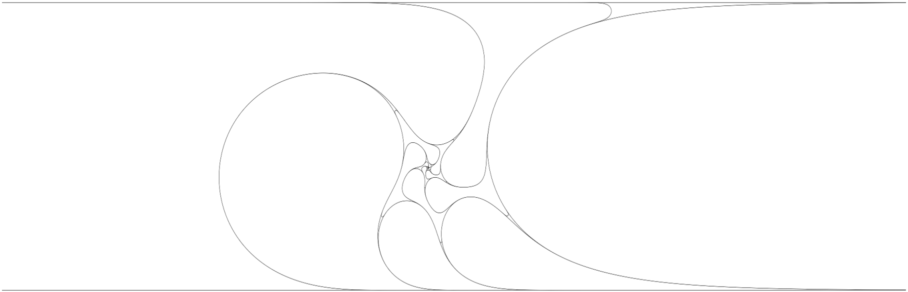

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It seems that γ_k^σ is "close" to $\overline{\Gamma^\sigma}$ for large k.

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Thus

$$\Gamma^\sigma \subset \overline{\Gamma^\sigma}$$

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<u>We need to show</u>: $\Gamma^\sigma \subset U$

- Take $z_0 \in \Gamma^\sigma \cap U$. $\implies \exists k_0 \in \mathbb{N}_0 \; \forall k \geq k_0 : \gamma_k^\sigma \cap U \neq \emptyset$
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Thus

$$\Gamma^\sigma \subset \overline{\Gamma^\sigma} = \overline{\bigcup_{k \geq k_0} \gamma_k^\sigma}$$

Lemma

Let $X \subset \hat{\mathbb{C}}$. Suppose for all $U \subset \mathbb{C}$ open we have

$$(X \cap U \neq \emptyset \quad \wedge \quad X \cap \partial U = \emptyset) \implies X \subset U.$$

Then X is connected.

Lemma (Connectivity of Γ^σ)

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Thus $\mathcal{I}(f)$ is connected for all $a \in (-1, \infty)$, which is our Theorem 1.

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Until now it is not known whether every $a \in \mathcal{J}(f_a)$ is accessible or not.

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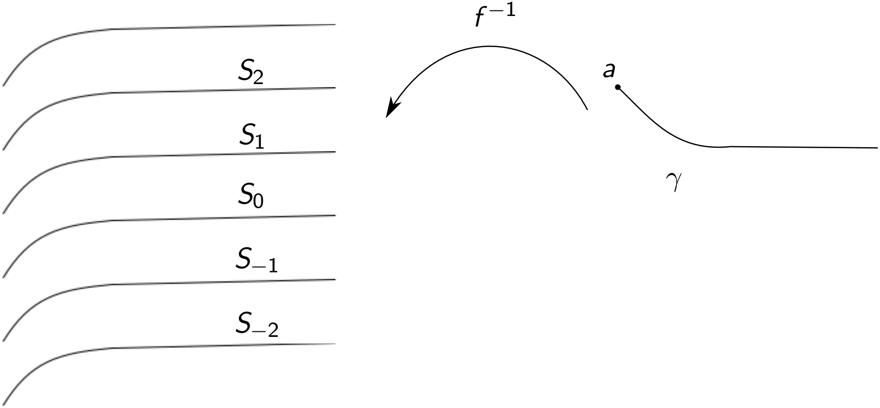

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Let $a \in \mathbb{C}$. If

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then $\mathcal{I}(f_a)$ is connected.

Thank you for your attention!