Mandelbrot and Sierpinski arcs and spirals

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Outline

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The Rational Map

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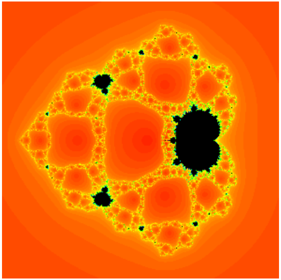
- Let's take a look, for $n = 2, d = 3$.

The parameter plane for $n = 2, d = 3$



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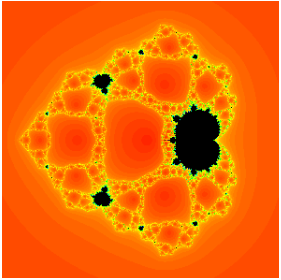

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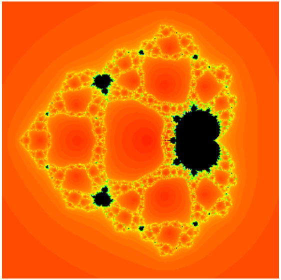

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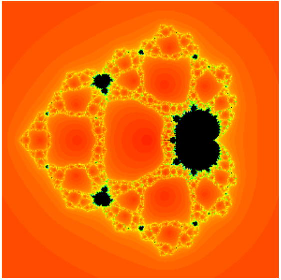
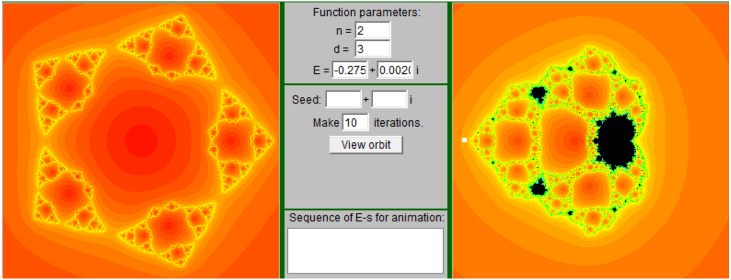


- Some program drew this parameter plane arranged into differently colored regions. What do they correspond to?
- The parameter plane has $Re(\lambda)$ and $Im(\lambda)$ for axes, and different values of λ probably result in different behavior for F_λ.
- Let's see what the dynamical plane looks like as we fix λ at different values:

λ outside the spaceship thing



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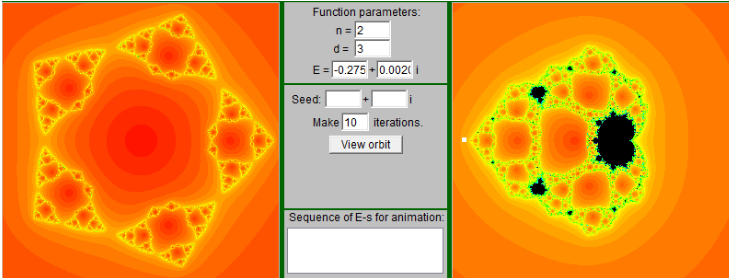
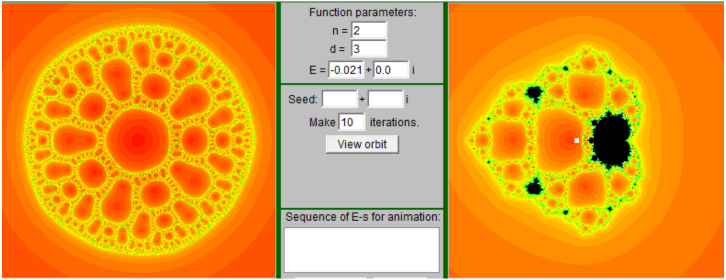


- That's kind of neat

λ in an orange region



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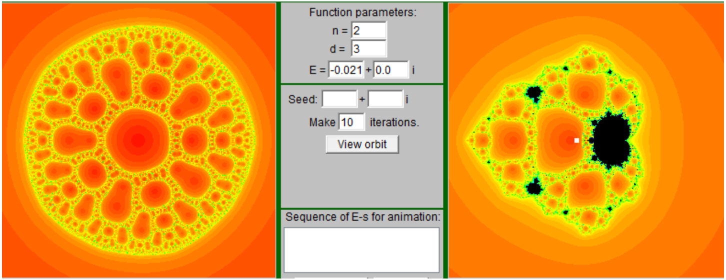
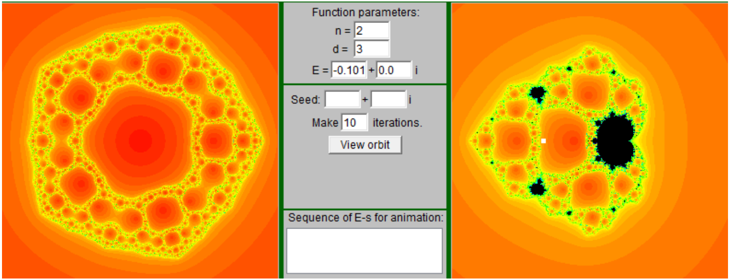


- Pretty cool.

λ in the same orange region



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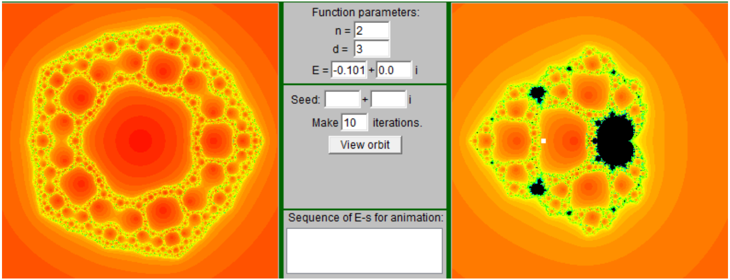

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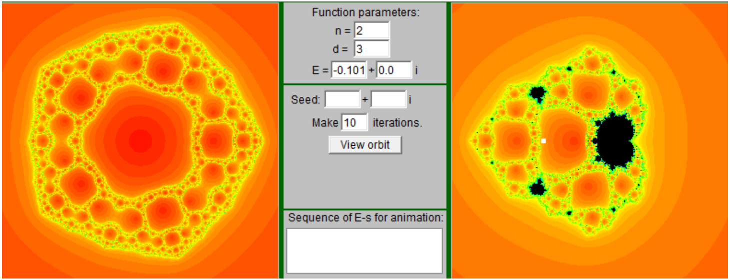

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- We could say something about topological conjugacy

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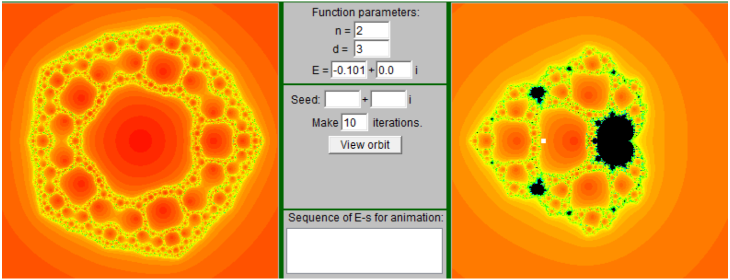
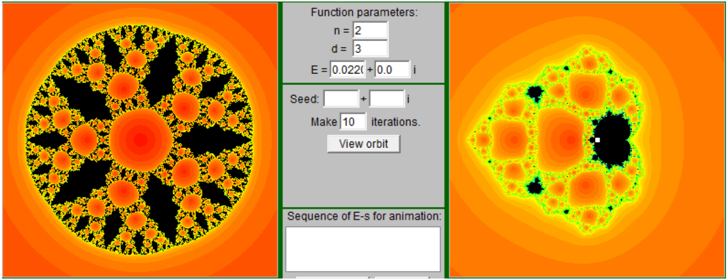


- That looks more or less the same.
- We could say something about topological conjugacy
- but we won't.

λ in a black spot



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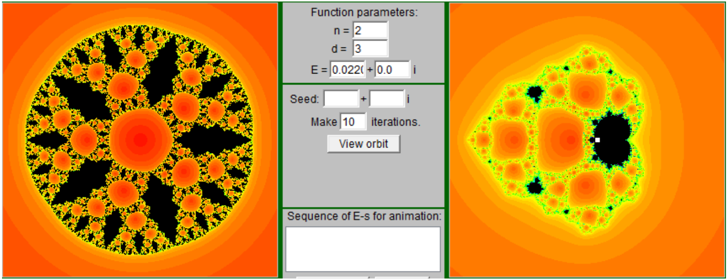

- What is happening here?

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- Such a way exists, based on the Julia set and orbits of the critical values of F_λ.
- Apologies if the following is boring to you:

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- There is a pole at the origin, so there is a neighborhood of the origin that is mapped into B_λ. If the preimage of B_λ surrounding the origin is disjoint from B_λ, we call this region the trap door and denote it by T_λ.

More Terminology

- The Julia set of F_λ, denoted $\mathcal{J}(F_\lambda)$, has several equivalent definitions. $\mathcal{J}(F_\lambda)$ is the set of all points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. Equivalently, $\mathcal{J}(F_\lambda)$ is the closure of the set of repelling periodic points of F_λ, and it is also the boundary of the set of points whose orbits tend to ∞ under iteration of F_λ.

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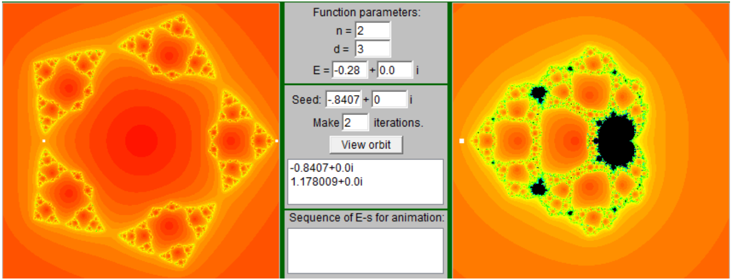
- For us, it's not that interesting.
- So we want to look at the behavior of the critical values of F_λ for different λ. The dynamical plane is symmetric under rotation, so it is enough to look at any one critical value to see the behavior of all of them.

Outline

Cantor set locus



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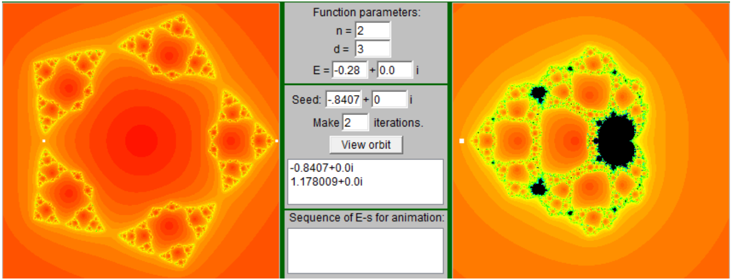

- v^λ lies in B_λ. In this case it is known that $\mathcal{J}(F_\lambda)$ is a Cantor set.

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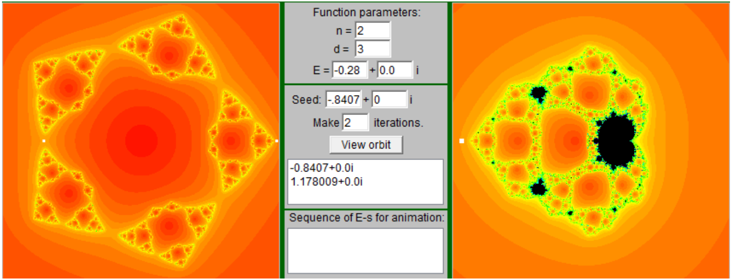
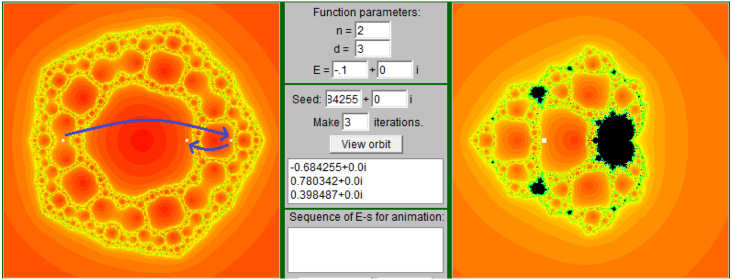


- v^λ lies in B_λ. In this case it is known that $\mathcal{J}(F_\lambda)$ is a Cantor set.
- The corresponding set of λ-values in the parameter plane is called the Cantor set locus.

Sierpinski holes



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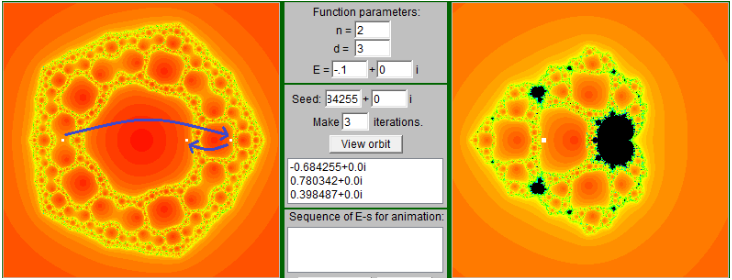

- v^λ enters T_λ at iteration 2 or higher. In this case it is known that $\mathcal{J}(F_\lambda)$ is a Sierpinski curve, i.e. a set that is homeomorphic to the Sierpinski carpet fractal.

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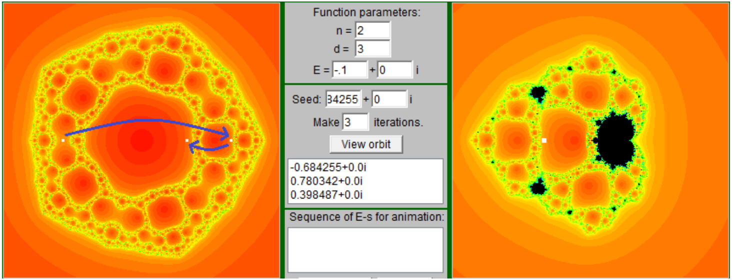
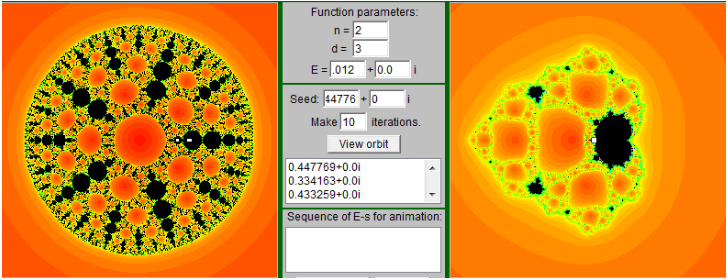


- v^{λ} enters T_{λ} at iteration 2 or higher. In this case it is known that $\mathcal{J}(F_{\lambda})$ is a Sierpinski curve, i.e. a set that is homeomorphic to the Sierpinski carpet fractal.
- The corresponding set of λ-values in the parameter plane are regions that we call Sierpinski holes.

The connectedness locus



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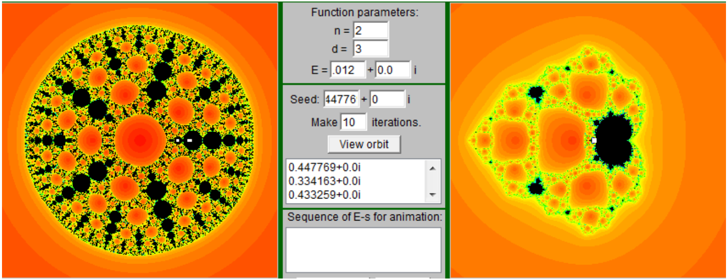

- v^{λ} does not escape to ∞.

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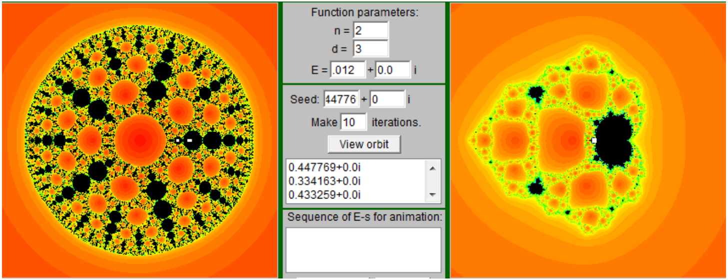
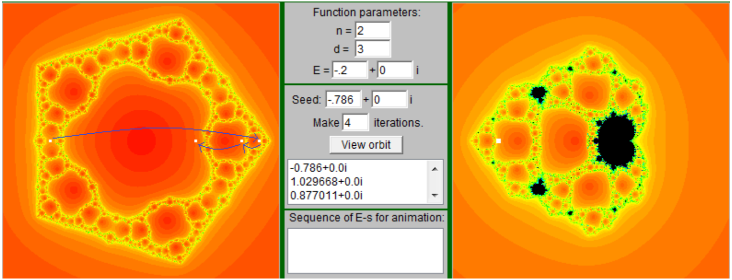


- v^λ does not escape to ∞.
- The corresponding set of λ-values in the parameter plane includes the Mandelbrot sets. Together with the Sierpinski holes, this region is called the connectedness locus, as $\mathcal{J}(F_\lambda)$ is a connected set for all λ in the locus.

Outline

Sierpinski hole of higher escape time



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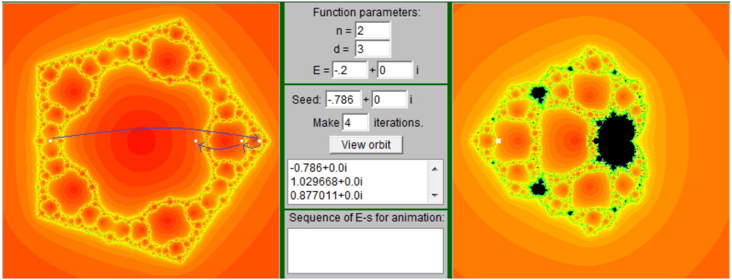
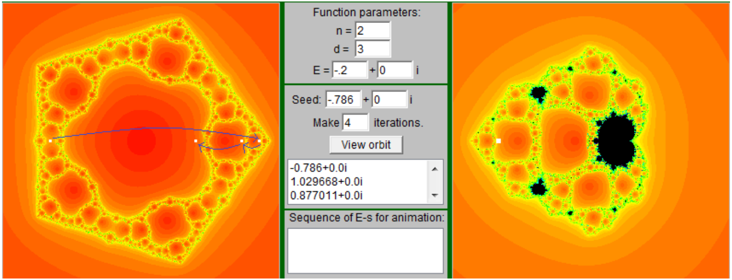


- For a λ in the next Sierpinski hole to the left:

Sierpinski hole of higher escape time



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- v^λ enters T_λ at iteration 3.

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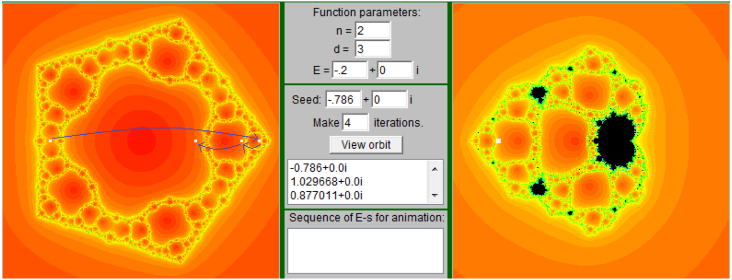
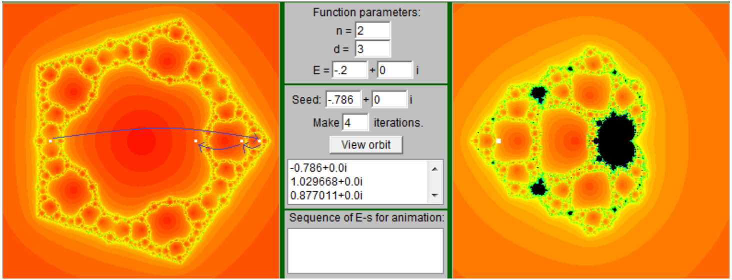


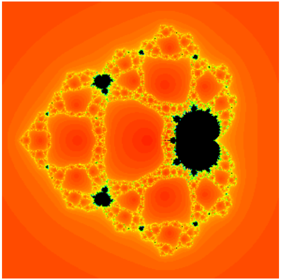
- For a λ in the next Sierpinski hole to the left:
- v^{λ} enters T_{λ} at iteration 3.
- The next Sierpinski hole along the negative real axis probably has escape time 4.

Sierpinski hole of higher escape time



- For a λ in the next Sierpinski hole to the left:
- v^{λ} enters T_{λ} at iteration 3.
- The next Sierpinski hole along the negative real axis probably has escape time 4.
- This idea of increasingly higher escape time Sierpinski holes might be interesting... let's look around some more.

More Mandelbrot sets



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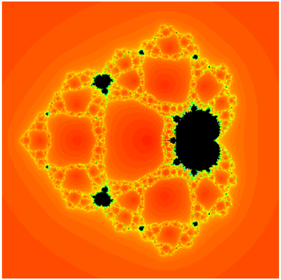

- There is the clearly visible principal Mandelbrot set.

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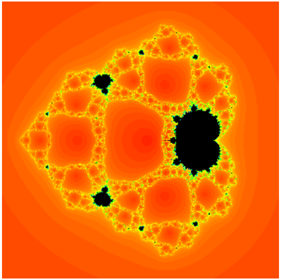

- There is the clearly visible principal Mandelbrot set.
- Also two baby Mandelbrot sets.

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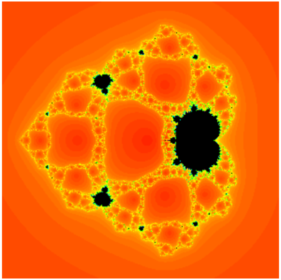
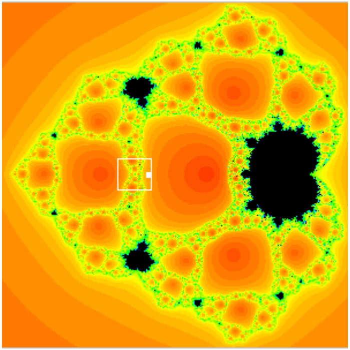


- There is the clearly visible principal Mandelbrot set.
- Also two baby Mandelbrot sets.
- Six more baby Mandelbrot sets. Are there others?

Zooming In



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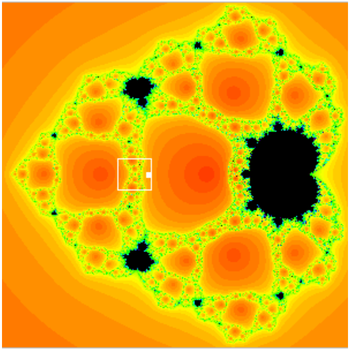

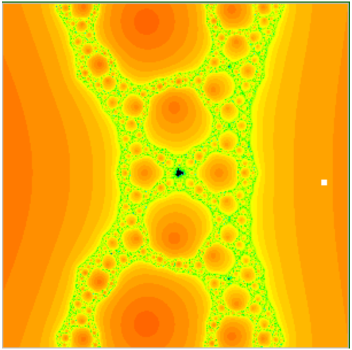

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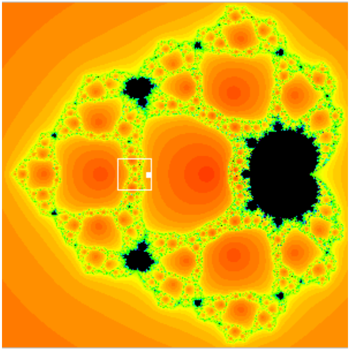
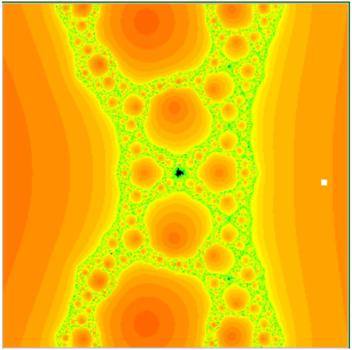
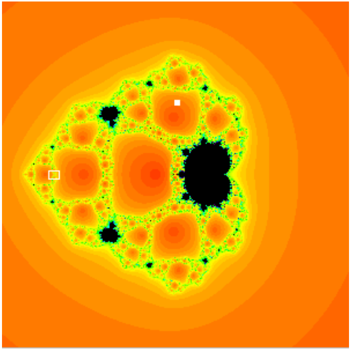




- There is one between the two Sierpinski holes.

Further along the negative real axis



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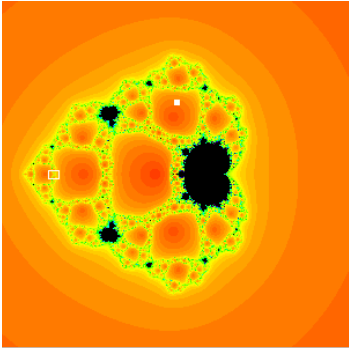

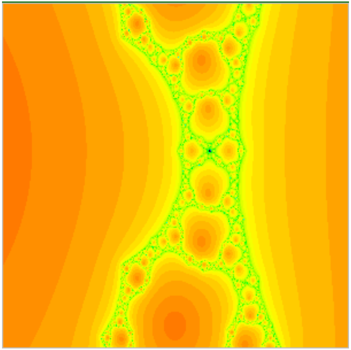

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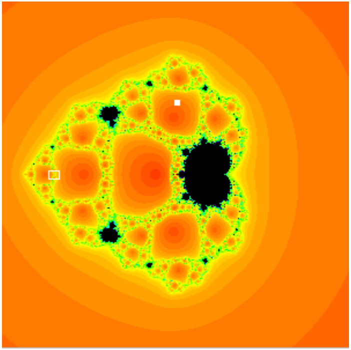
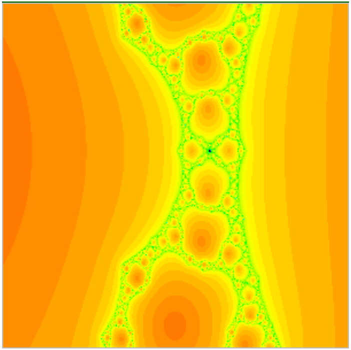
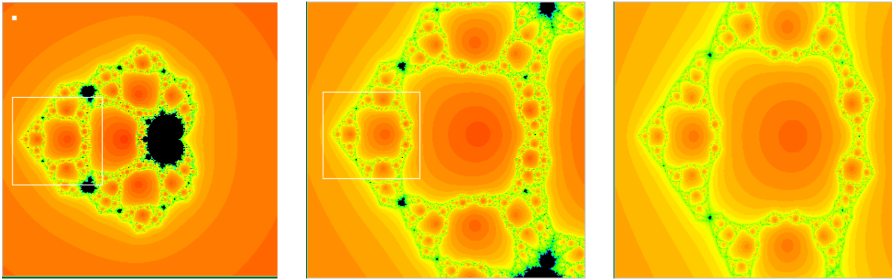




- Looks like another one between the next pair of Sierpinski holes. Is there a pattern?

Conjecture



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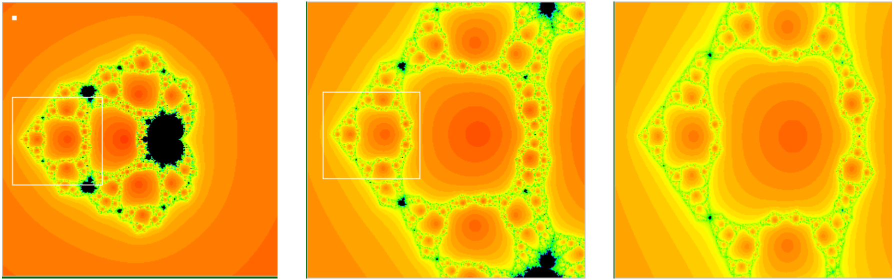

- There are infinitely many Sierpinski holes along the negative real axis of the parameter plane.

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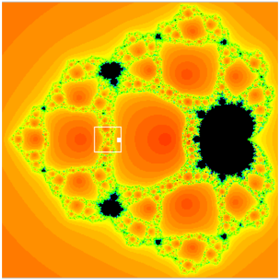
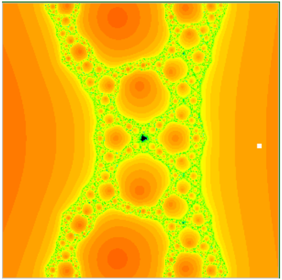
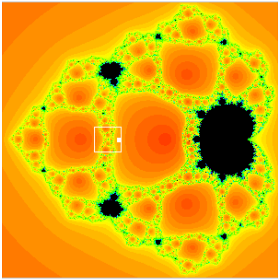
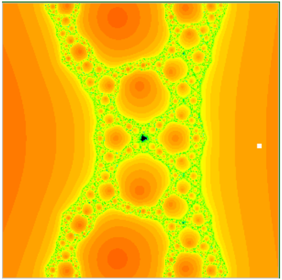




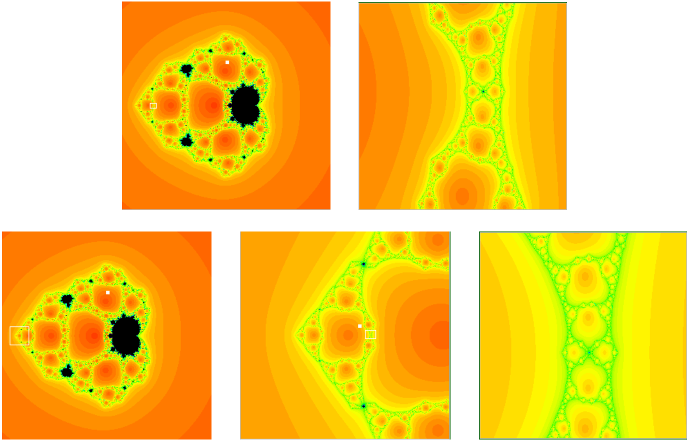
Conjecture





- Between each pair of Sierpinski holes is a Mandelbrot set, though it might be hard to see.

If it works the first 3 times, it works all the time



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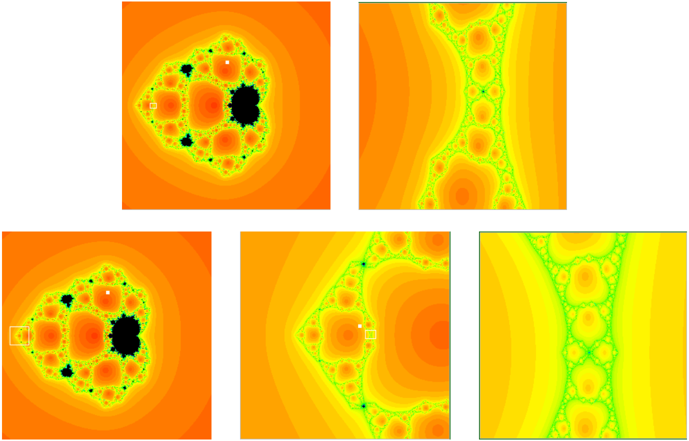
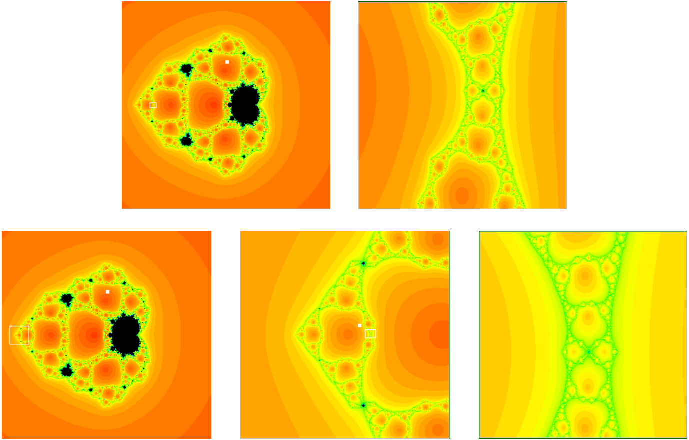


- We can't keep zooming in for each of the (infinite number of) Mandelbrot sets.

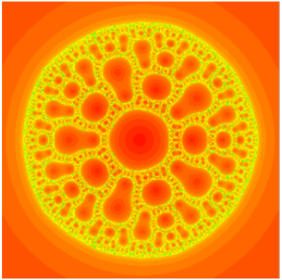
If it works the first 3 times, it works all the time



- Is there a way to prove the existence of this alternating arc of infinite Sierpinski holes and Mandelbrot sets using the dynamical plane?

Outline

The dynamical plane for $n = 2, d = 3$



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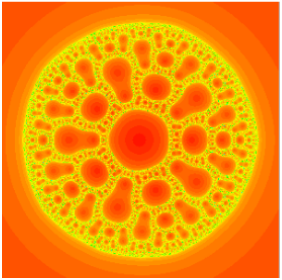

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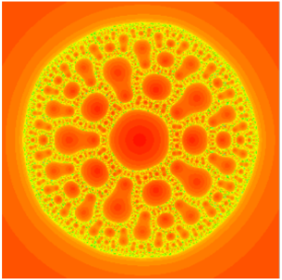

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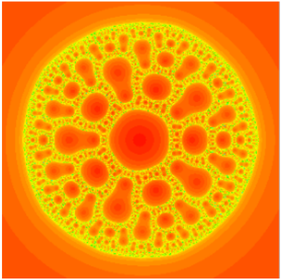
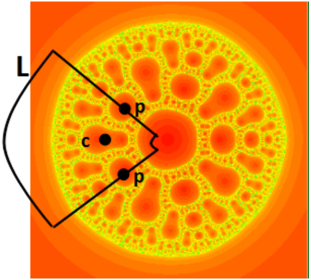


- This is the dynamical plane for $n = 2, d = 3$.
- To construct the objects in the Sierpinski Mandelbrot arc we will need to consider some closed sets in the dynamical plane.
- We will also restrict attention to an annular region in the parameter plane. The details aren't that important for this talk.

The left wedge



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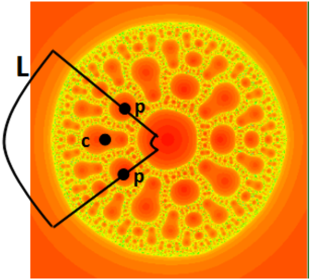

- Let L^λ be the closed portion of the wedge with inner boundary in the trapdoor, outer boundary in the basin, and straight line boundaries that are part of the two adjacent prepole rays as shown.

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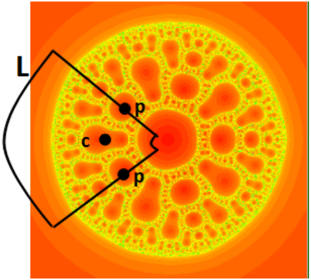

- Let L^λ be the closed portion of the wedge with inner boundary in the trapdoor, outer boundary in the basin, and straight line boundaries that are part of the two adjacent prepole rays as shown.
- There is one critical point is in the interior of L^λ.

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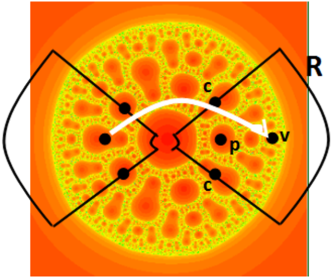

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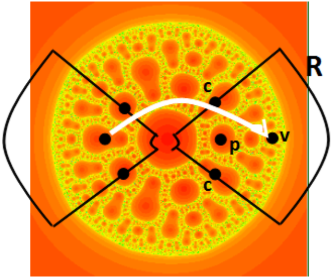

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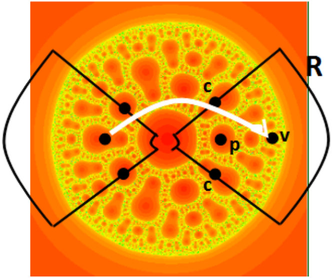
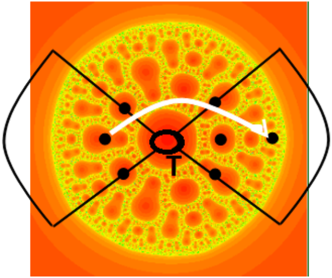


- Let R^λ be the symmetric right wedge. The straight line boundaries are part of two adjacent critical point rays.
- There is one prepole in the interior of R^λ.
- The critical value corresponding to the critical point in the interior of L^λ is in R^λ.

The subset of the trapdoor



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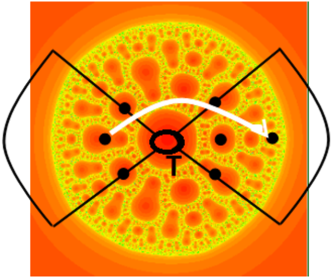

- Let T^λ be a closed subset of the trapdoor containing 0 such that $L^\lambda \cup T^\lambda \cup R^\lambda$ are connected, and they only intersect along boundaries.
- This union will be referred to informally as the bowtie.

Proposition

- There are more parts to the proposition for the paper in the works, but the part we care about for now is:

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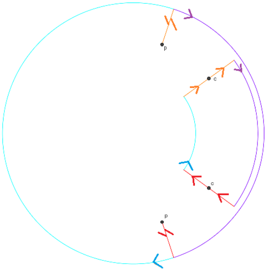
3. See part 2.

"Proof" of part 1



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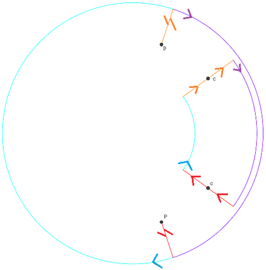

- The critical point rays map to the prepole rays.

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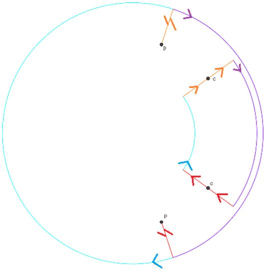

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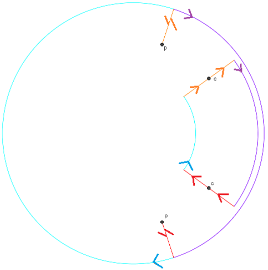

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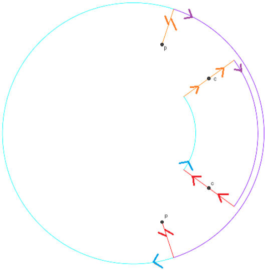

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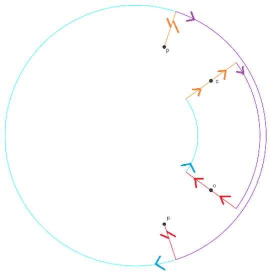
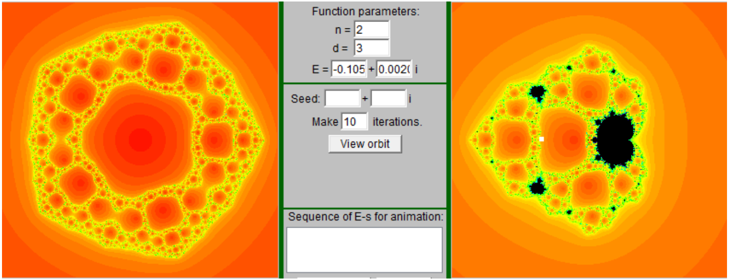


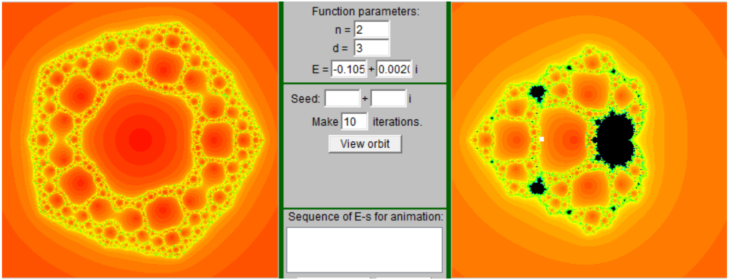
- The critical point rays map to the prepole rays. The boundary of R^λ in B^λ maps to the outer arc on the right. The boundary of R^λ in T^λ maps to the outer arc on the left.
- Then the image of R^λ properly contains the interiors of both R^λ and L^λ.
- In other words, inside R^λ is a bowtie which consists of a preimage of L^λ, a preimage of T^λ, and a preimage of R^λ.

Outline

Drawing a picture

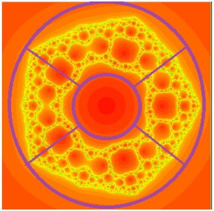


Drawing a picture



- Let's dress that dynamical plane up with a bowtie.

Bowties in bowties



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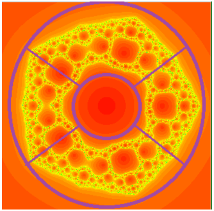

- That bowtie contains a preimage of R_λ, which should have another bowtie in it. Which should have another bowtie in it. Which should have another bowtie in it.

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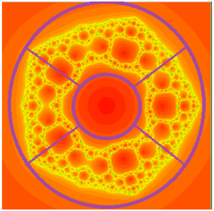

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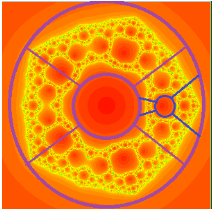

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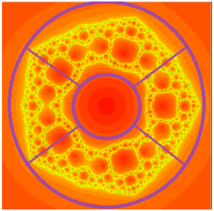

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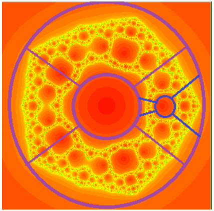

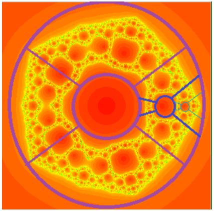

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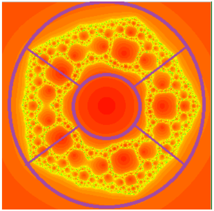

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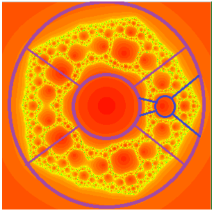

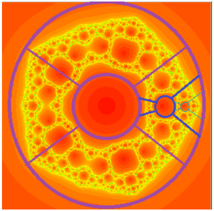

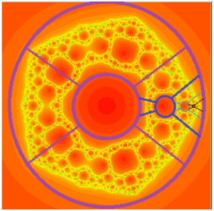

Back to the parameter plane

- It turns out each preimage of L^λ and each preimage of T^λ corresponds to a Mandelbrot set and a Sierpinski hole, respectively, in the parameter plane.

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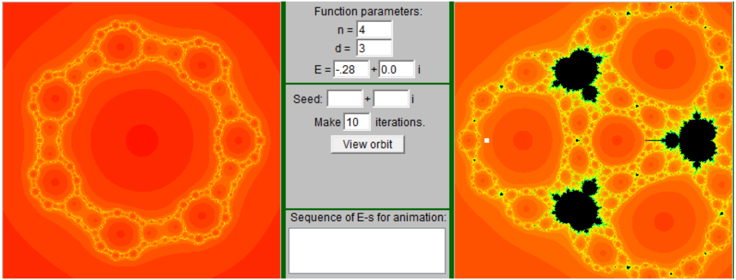
- There's definitely more going on for this function, but I want to talk about the case where $n = 4, d = 3$.

Outline

The parameter plane for $n = 4, d = 3$



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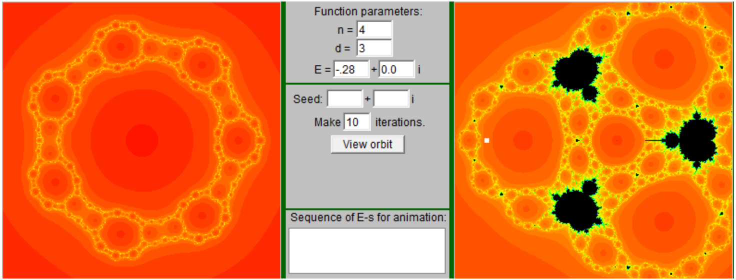

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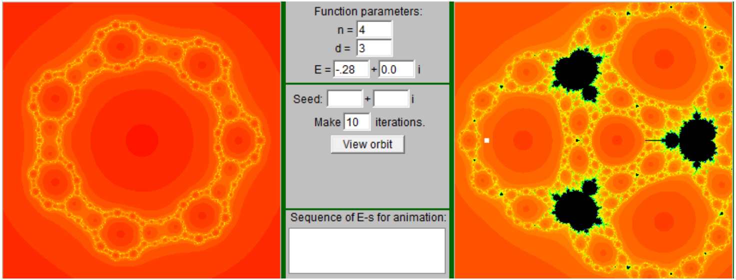

- Looks like we still have a set of infinitely many alternating Sierpinski holes and Mandelbrot sets along the negative real axis.
- We should be able to prove that by a similar argument to $n = 2, d = 3$.

By analogy

- There are now 7 critical points, critical values, and prepoles.

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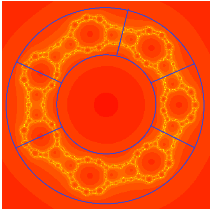

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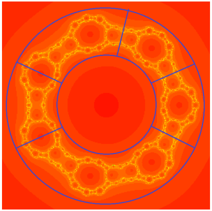

- In fact, we can make another right wedge above the existing one. Let's refer to the wedge symmetric to L^λ as R_0^λ, and to the new one as R_1^λ. Looks like our bowtie is now lopsided.

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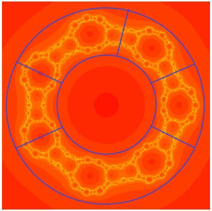

- In fact, we can make another right wedge above the existing one. Let's refer to the wedge symmetric to L^λ as R_0^λ, and to the new one as R_1^λ. Looks like our bowtie is now lopsided.
- The existence of Mandelbrot sets and Sierpinski holes based on sets in the dynamical plane depends on being able to vary $\mathrm{Arg}(\lambda)$ by a certain amount, and adding more right wedges in the $n = 2, d = 3$ case would have violated that.

(Lopsided) bowties in (lopsided) bowties

- It's reasonable to assume that R_0^λ contains a (now lopsided) bowtie as before.

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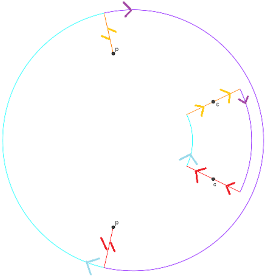

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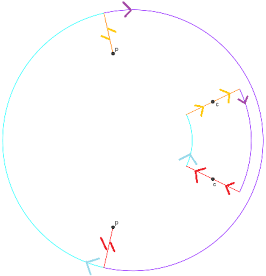

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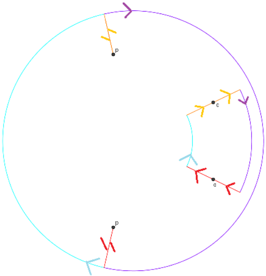

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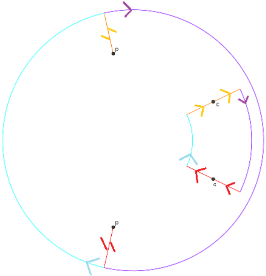

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- What about R_1^λ?

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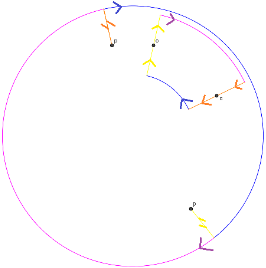

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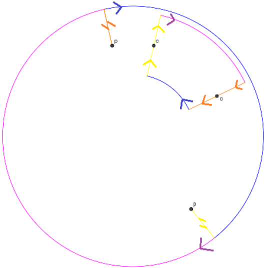
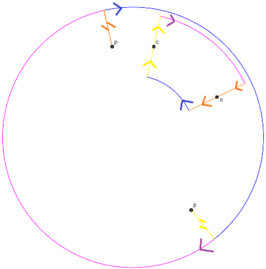


- So inside R_0^λ is a bowtie which consists of a preimage of L^λ, a preimage of T^λ, a preimage of R_0^λ, and a preimage of R_1^λ.

Bowties in bowties



- So inside R_0^λ is a bowtie which consists of a preimage of L^λ, a preimage of T^λ, a preimage of R_0^λ, and a preimage of R_1^λ.
- Inside R_1^λ is a bowtie which consists of a preimage of L^λ, a preimage of T^λ, a preimage of R_0^λ, and a preimage of R_1^λ, but "rotated" π radians before being placed in R_1^λ. Note the orientation is preserved.

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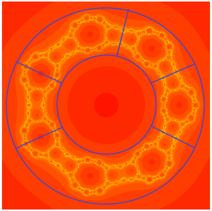

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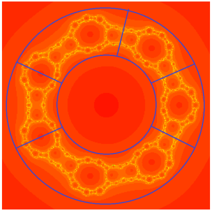

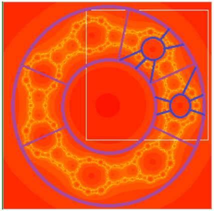

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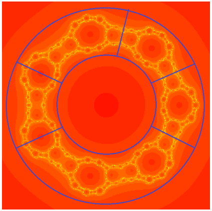

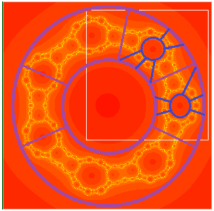

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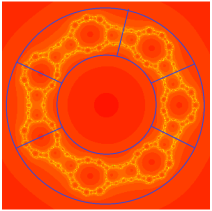

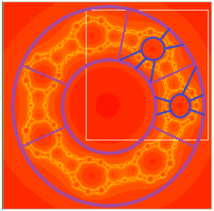

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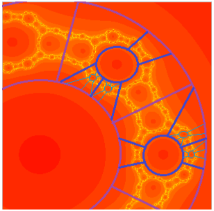

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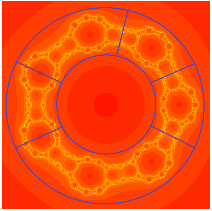 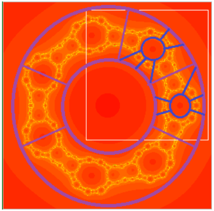

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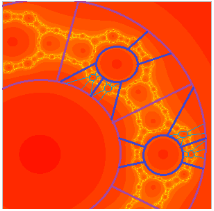 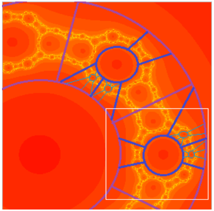

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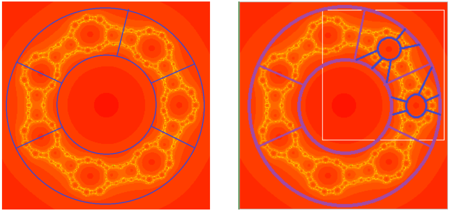

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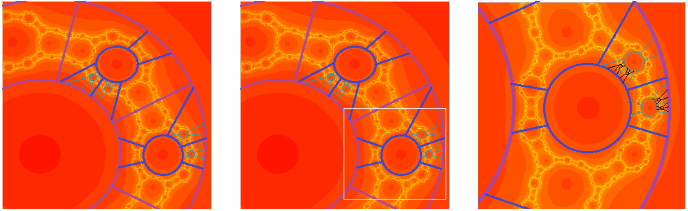

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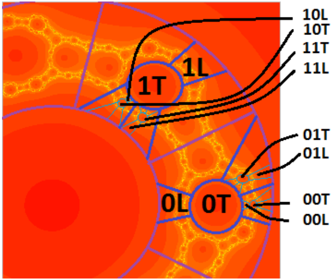

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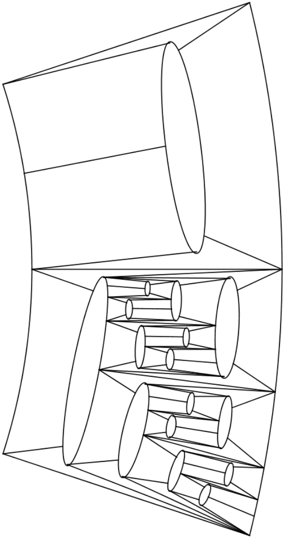
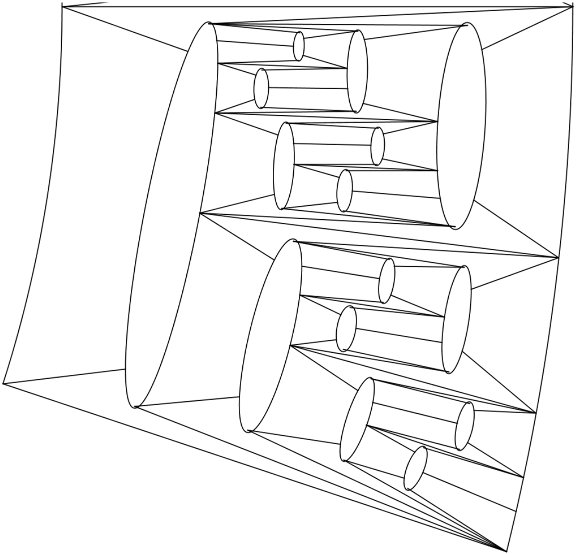
- A more stylized depiction of the two wedges would then look like:

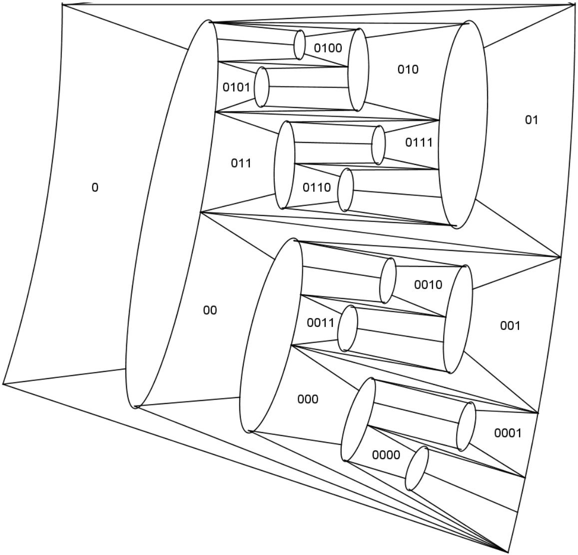
Stylized depiction of the right wedges



Stylized depiction of R_0^λ



Stylized depiction of R_0^λ with labeled L preimages



Outline

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- You get the idea.

Sequences ending in 1

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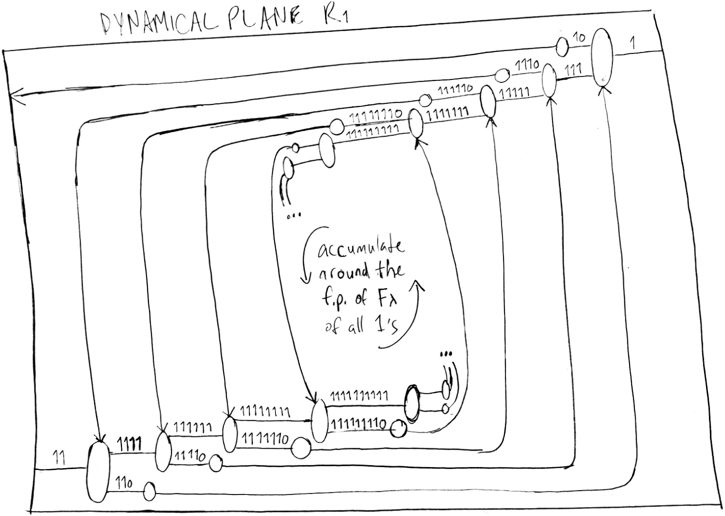

More than one spiral

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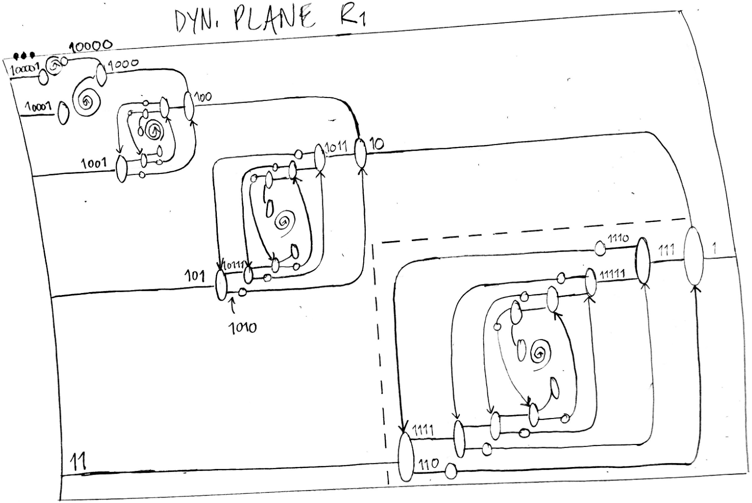

How many?

- This suggests that each element in the set $\{1, 10, 100, 1000, ...\}$ has a unique spiral in the dynamical plane that corresponds to a spiral in the parameter plane consisting of infinitely many arcs of infinitely many alternating S-holes and M-sets that accumulate in infinitely many S-holes along the spiral.

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- Since each element in that set has a preimage in R_0^λ, there are that many corresponding spirals in R_0^λ as well!

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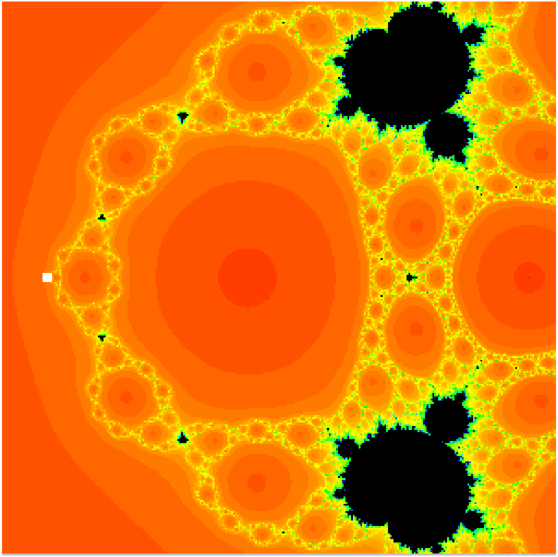
- For that matter, where are the spirals?
- That's a good question...

Speculation



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What's next?

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- Are there other structures?
- What happens when we increase to $n = 6$? $d = 5$? Can we generalize?

Thanks!

I had a great time here! Happy turkey day!

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Long Live Catalonia!

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