Tongues and bifurcations on a family of degree 4 Blaschke products

Jordi Canela
Institute of Mathematics Polish Academy of Sciences
IMPAN

— Joint work with: **Núria Fagella** and **Antonio Garijo** —

Barcelona, 23 November 2015

Degree 4 Blaschke products

We want to study the degree 4 Blaschke products

$$B_a(z) = z^3 \frac{z-a}{1-\bar{a}z}.$$

They are rational perturbations of the doubling map of the circle

$$R_2(z) = z^2, \text{ equivalently given by } \theta \to 2\theta \ (\text{mod } 1).$$

These products are the rational version of the double standard map:

	$\mathbb{S}^1 \to \mathbb{S}^1$	$\mathbb{C}^* \to \mathbb{C}^*$	$\hat{\mathbb{C}} \to \hat{\mathbb{C}}$
Standard map	$\theta \to \theta + \alpha + \beta \sin \theta$	$e^{i\alpha} \cdot z \cdot e^{\beta/2(z+1/z)}$	$e^{it} z^2 \frac{z-a}{1-\bar{a}z}$
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M. Herman, *Sur la conjugaison des difféomorphismes du cercle à des rotations*, 1976
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Properties of these Blaschke products

The main properties of these Blaschke products are the following:

- They leave $\mathbb{S}^1$ invariant.
- They are symmetric with respect to $\mathbb{S}^1$, i.e., $B_a(z) = (B_a(z^*))^*$, where $z^* = 1/\overline{z}$.
- $z = 0$ and $z = \infty$ are superattracting fixed points of local degree 3.
- $z_\infty = 1/\overline{a}$ and $z_0 = a$ are the only pole and zero respectively.

They have two "free" critical points $c_\pm$ (i.e., $B_a'(c_\pm) = 0$)

$$c_\pm = a \cdot \frac{1}{3|a|^2}\left(2 + |a|^2 \pm \sqrt{(|a|^2 - 4)(|a|^2 - 1)}\right)$$

which control all possible stable dynamics other than the basins of attraction of $z = 0$ and $z = \infty$.

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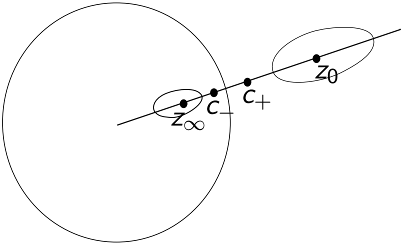
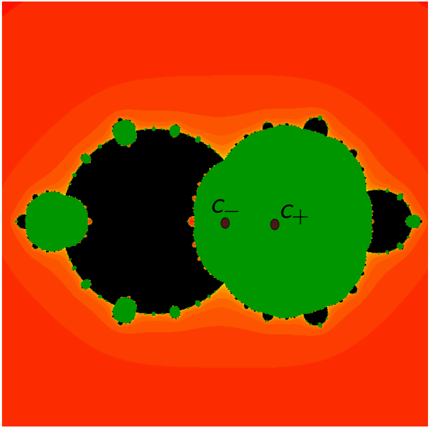
We study the dynamics of these products depending on $|a|$.

Case $|a| > 2$

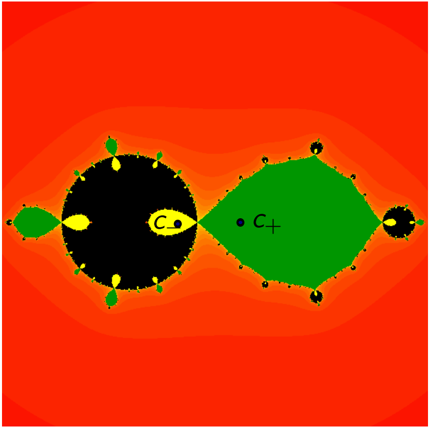
$z_\infty, c_- \in \mathbb{D}$
$z_0, c_+ \in \mathbb{C} \setminus \overline{\mathbb{D}}$
$B_a|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ is a covering of degree 2.



$c_- = 1/\overline{c_+} \Rightarrow$ Critical orbits are symmetric w.r.t. $\mathbb{S}^1$.



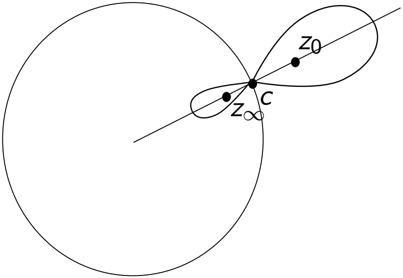
Dynamical plane of $B_{5/2}$. Both free critical orbits accumulate on a fixed point in $\mathbb{S}^1$.



Dynamical plane of B_4. Each critical orbit accumulates on a different attracting cycle.

Case $|a| = 2$

$c_+ = c_- = a/2 \in \mathbb{S}^1$
$z_\infty \in \mathbb{D}$
$z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$



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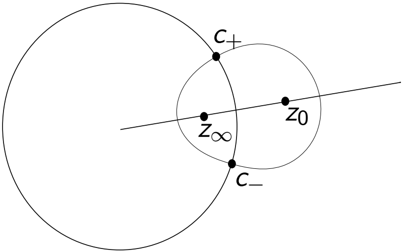
Case $1 < |a| < 2$

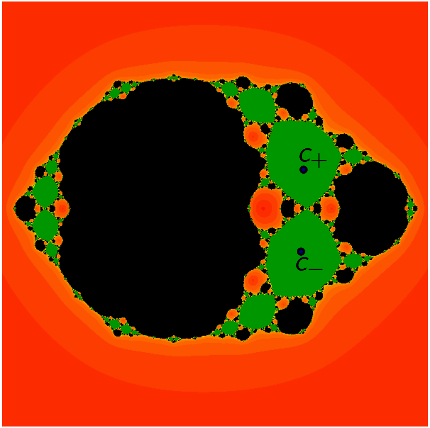
There are two different critical points:

$$c_+ = a \cdot \frac{1}{3|a|^2}\left(2 + |a|^2 + i\sqrt{(4 - |a|^2)(|a|^2 - 1)}\right) = a \cdot k$$

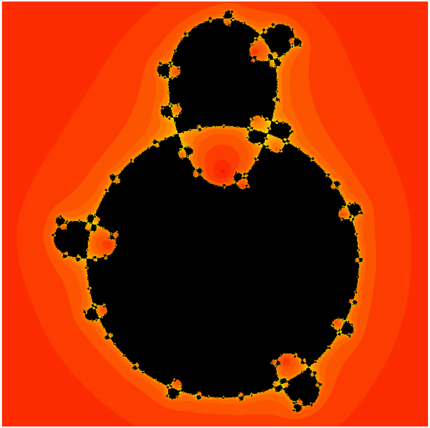
$$c_- = a \cdot \bar{k}$$

The critical points satisfy $|c_\pm| = 1$.
The critical orbits are not symmetric.

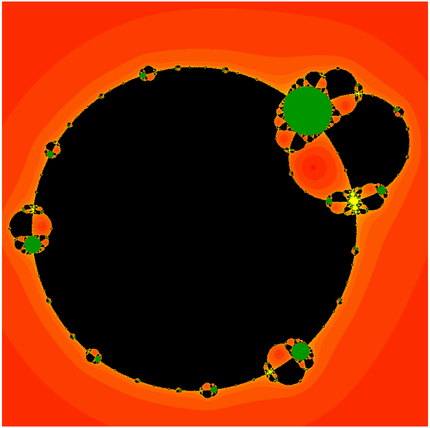


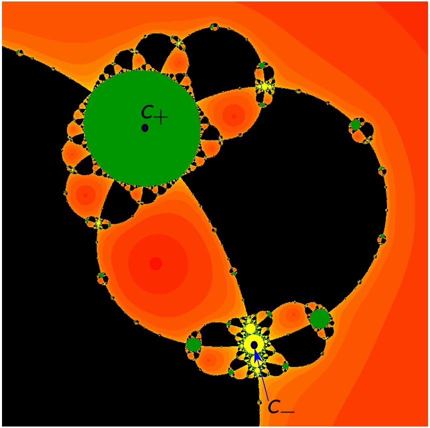


Dynamical plane of $B_{3/2}$. We see in green an attracting basin of a period 2 cycle.



Dynamical plane of $B_{3/2i}$. There are no other attracting basins than the ones of $z = 0$ and $z = \infty$.

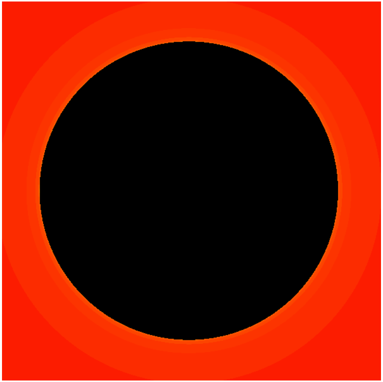


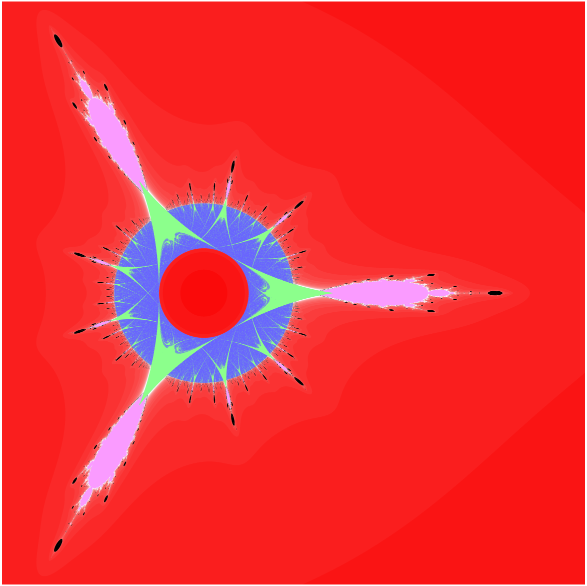


Dynamical plane of the Blaschke product $B_{1,07398+0,5579i}$.

Case $|a| \leq 1$

If $|a| \leq 1$ then $B_a(\mathbb{D}) = \mathbb{D}$ and the dynamics is well understood.





Parameter plane of B_a. It has been drawn by iterating the critical point c_+.

Remark: B_a and $B_{\xi a}$ are conjugate, where ξ is a third root of unity.

Tongues of the Blaschke family

Let a s.t. $|a| \geq 2$. Then, $B_{a|\mathbb{S}^1}$ is a degree 2 covering of $\mathbb{S}^1$.

$B_{a|\mathbb{S}^1}$ is semiconjugate to the doubling map $\theta \to 2\theta \pmod 1$ by a unique continuous map H_a.

H_a sends periodic points to periodic points of the same period.

Definition

We say that a, $|a| \geq 2$, is of type τ if $B_a|_{\mathbb{S}^1}$ has an attracting cycle and $H_a(x_0) = \tau$, where x_0 is the marked point of the attracting cycle.

Definition

We define the tongue $T_\tau = \{a \mid 2 \leq |a|, a \text{ has type } \tau\}$.

Remark

Since H_a sends periodic points to periodic points, any realizable type $\tau \in \mathbb{S}^1$ is a periodic point of the doubling map $\theta \to 2\theta \pmod 1$.

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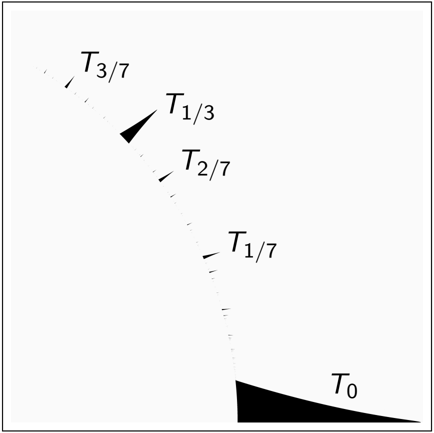

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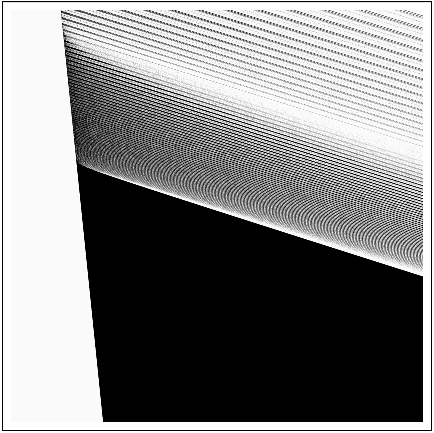


Zoom in the tongues

The left figure shows the tongues of the Blaschke family for $a = re^{2\pi i\alpha}$ such that $0 < \alpha < 1/6$. The right figure is a zoom near the boundary of the fixed tongue T_0.

Theorem

Given any periodic point τ of the doubling map the following results hold.

(a) *The tongue T_τ is not empty and consists of three connected components (only one connected component if we consider the parameter plane modulo the symmetries given by the third roots of the unity).*

(b) *Each connected component of T_τ contains a unique parameter r_τ, called the root of the tongue, such that B_{r_τ} has a superattracting cycle in $\mathbb{S}^1$. The root r_τ satisfies $|r_\tau| = 2$.*

(c) *Every connected component of T_τ is simply connected.*

(d) *The boundary of every connected component of T_τ consists of two curves which are continuous graphs as function of $|a|$ and intersect each other in a unique parameter a_τ called the tip of the tongue.*

M. Misiurewicz and A. Rodrigues, *Double standard maps*, 2007

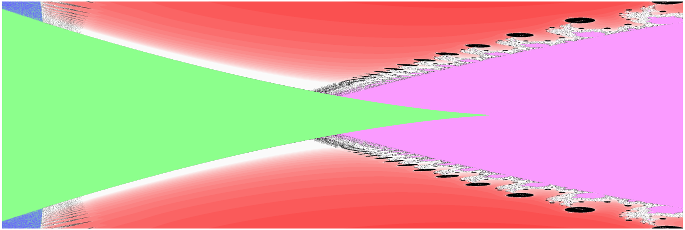
A. Dezotti, *Connectedness of the Arnold tongues for double standard maps*, 2010

Bifurcations around the tip of the tongues

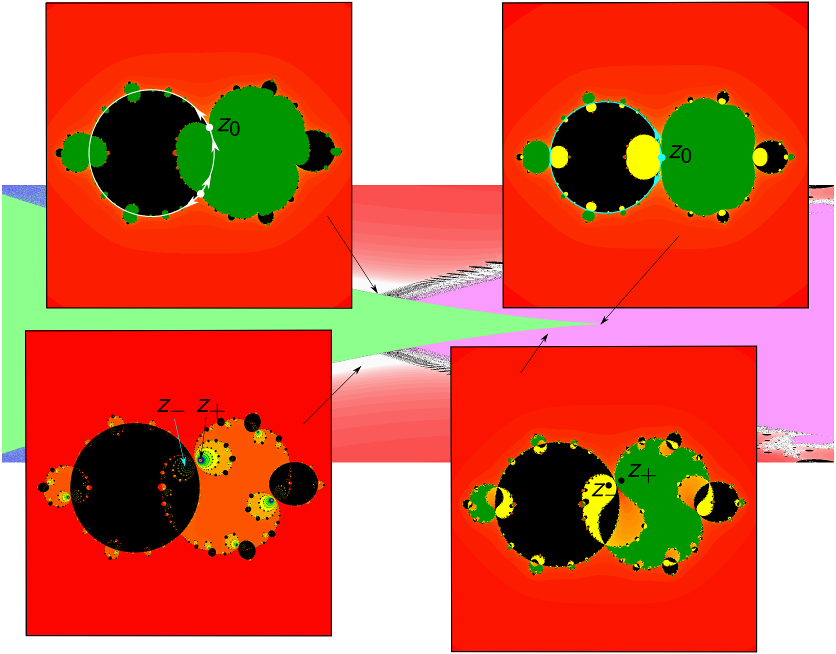
Theorem

Given any tongue T_τ, there exists a neighborhood U of the tip of the tongue in which only one of the following can occur:

- $a \in T_\tau \Rightarrow B_{a|\mathbb{S}^1}$ *has an attracting periodic cycle.*
- $a \in \partial T_\tau$ *and* B_a *has a parabolic periodic cycle in* $\mathbb{S}^1$.
- $a \notin \overline{T_\tau}$ *and* B_a *has two different attracting periodic cycles outside* $\mathbb{S}^1$.



M. Misiurewicz and R. A. Pérez , *Real saddle-node bifurcation from a complex point of view*, 2008.


z_0
z_0
z_- z_+
z_+
z_-

Tools used in the proof

The result is based on the holomorphic index of the fixed points of B_a^n. We use the following:

(a) If z_0 is a fixed point of B_a of multiplier $\rho \neq 1$ then $i(z_0) = 1/(1-\rho)$.

(b) If m different fixed points collide in a parabolic point z_0 of multiplier 1, their indexes tend to infinity, even if the sum of their indexes tends to the finite index $i(z_0)$ of the parabolic point.

Moreover, we cannot have a curve of "tip" parameters.

Lemma

For fixed $n > 0$, there is only a finite number of parameters $a \in \mathbb{C}$ for which the Blaschke product B_a has a parabolic cycle of exact period n, multiplier 1 and multiplicity 3.

J. H. Hubbard, D. Schleicher, *Multicorns are not path connected*, 2012

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Finitely many tip parameters

We embed the Blaschke products B_a into the more general almost bicritical rational family

$$G_{a,b}(z) = bz^3 \frac{z-a}{1-az},$$

where $a, b \in \mathbb{C}$. We prove the lemma for the family $G_{a,b}$.

The parameters (a, b) such that $G_{a,b}$ has a parabolic cycle multiplier 1 and multiplicity 3 are given by

$$\begin{cases} G^n_{a,b}(z) = z, \\ (\frac{\partial}{\partial z} G^n_{a,b})(z) = 1, \\ (\frac{\partial^2}{\partial z^2} G^n_{a,b})(z) = 0, \end{cases} \quad (1)$$

and

$$G^m_{a,b}(z) \neq z \quad \text{for all} \quad m < n. \quad (2)$$

The set of solutions of (1) and (2) is a quasiprojective variety, say Y.

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The projection of Y over the variable a is bounded by next lemma.

Lemma

The non-escaping set of the family $G_{a,b}$ is bounded on the parameter a.

It follows from Chevalley's Theorem that the projection of Y over the variable a is finite since constructible sets in $\mathbb{C}$ are either finite or dense.

Theorem (Chevalley's Theorem)

Any morphism of quasiprojective varieties sends constructible sets to constructible sets.

Finally, we have

Lemma

For fixed $a_0 \in \mathbb{C}$, the non-escaping set of the family $G_{a_0,b}$ is bounded on the parameter b.

We conclude that the projection of Y over the variable b is also finite.

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It follows from Chevalley's Theorem that the projection of Y over the variable a is finite since constructible sets in $\mathbb{C}$ are either finite or dense.

Theorem (Chevalley's Theorem)

Any morphism of quasiprojective varieties sends constructible sets to constructible sets.

Finally, we have

Lemma

For fixed $a_0 \in \mathbb{C}$, the non-escaping set of the family $G_{a_0,b}$ is bounded on the parameter b.

We conclude that the projection of Y over the variable b is also finite.

The projection of Y over the variable a is bounded by next lemma.

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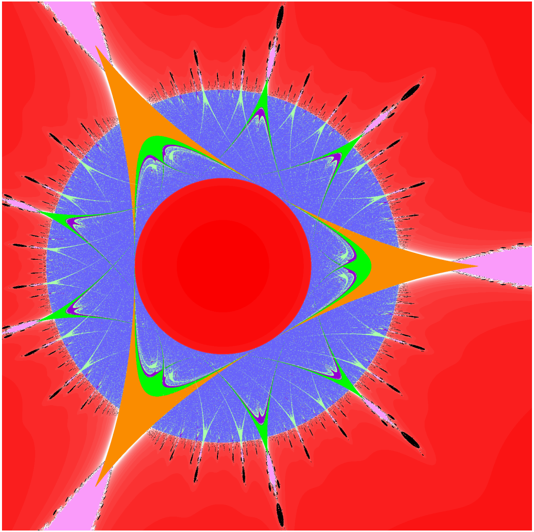
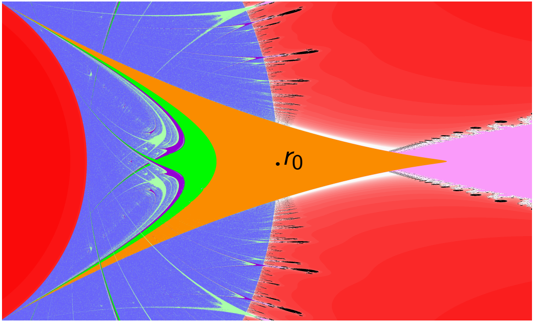
Extending the tongues



Definition

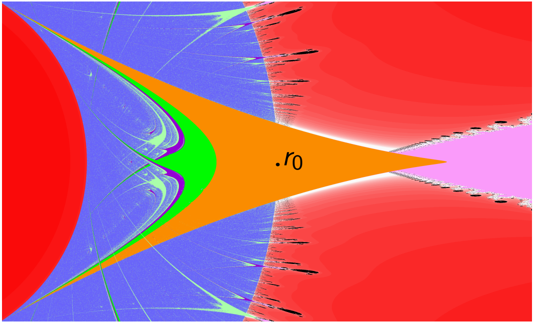
The *extended tongue* ET_τ is defined to be the set of parameters for which the attracting cycle of T_τ can be continued.

Notice that, since there are two different critical points moving independently for $1 < |a| < 2$, two different tongues may intersect each other.



Theorem

The boundary of the extended fixed tongue ET_0 consists of an exterior component of parameters with multiplier 1 and an interior component of parameters with multiplier -1. A period doubling bifurcation takes place along the inner boundary.



Thank you for your attention!