Rotation Sets and Complex Dynamics

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A degree $d \geq 2$ rotation subset of the circle $\mathbb{R}/\mathbb{Z}$ is a compact set on which the multiplication by d map $t \mapsto d \cdot t \pmod{\mathbb{Z}}$ acts in an order-preserving fashion and therefore has a well-defined rotation number. The study of rotation sets with rational rotation number was initiated in the early 1990's by Goldberg and Milnor in their work on fixed point portraits of complex polynomials, and thrived in the following two decades. In the irrational case, however, the theory did not see adequate development, especially for $d \geq 3$.

In the first part of this series of lectures, we present a systematic treatment of the abstract theory of rotation sets in both rational and irrational cases. In particular, we give a new unified approach to the Goldberg-Tresser theorem according to which minimal rotation sets are uniquely determined by their rotation number and deployment probability vector. Applications of this result in understanding the metric structure of rotation sets and explicit computations will be discussed.

In the second part, we explore the link between rotation sets and complex polynomial maps, emphasizing the irrational cases of low degree. We outline how rotation sets occur in the dynamical plane of polynomials, and how parameter spaces provide a concrete catalog of all rotation sets. This includes a quick recap of the well-known connection between degree 2 rotation sets and the quadratic family, as well as new and surprising connections between degree 3 rotation sets and the cubic family.