On the measure of the escaping set of a quasiregular analogue of sine

Sebastian Vogel

Christian-Albrechts-Universität zu Kiel

Barcelona, 14 June 2013

1. Motivation

We are interested in a natural generalisation of holomorphic functions to higher dimensions.

We are interested in a natural generalisation of holomorphic functions to higher dimensions.
A function $f : G \to \mathbb{C}$ is holomorphic, if and only if

- f is C^1 in the real sense and
- $||Df(z)||^2 = J_f(z)$ for all $z \in G$,

where J_f denotes the Jacobian of f.

We are interested in a natural generalisation of holomorphic functions to higher dimensions.
A function $f : G \to \mathbb{C}$ is holomorphic, if and only if

- f is C^1 in the real sense and
- $||Df(z)||^2 = J_f(z)$ for all $z \in G$,

where J_f denotes the Jacobian of f. Hence holomorphic functions map infinitesimal small circles to infinitesimal circles.

We are interested in a natural generalisation of holomorphic functions to higher dimensions.
A function $f : G \to \mathbb{C}$ is holomorphic, if and only if

- f is C^1 in the real sense and
- $||Df(z)||^2 = J_f(z)$ for all $z \in G$,

where J_f denotes the Jacobian of f. Hence holomorphic functions map infinitesimal small circles to infinitesimal circles.
We can easily generalise this definition to higher dimensions. But if we take $G \subset \mathbb{R}^d$ open and $f : G \to \mathbb{R}^d$ satisfying

- f is C^1 in the real sense and
- $||Df(x)||^d = J_f(x)$ for all $x \in G$,

then f is either constant or a sense preserving Möbius transformation.

In order to get a more interesting class of functions, we relax the conditions in the following way:

In order to get a more interesting class of functions, we relax the conditions in the following way:

Definition

A continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$ is called *quasiregular*, if

- $f \in W^1_{d,loc}(\mathbb{R}^d)$
- and $K_1 \geq 1$ exists, such that $||Df(x)||^d \leq K_1 J_f(x)$ a.e.,

where $W^1_{d,loc}(\mathbb{R}^d)$ denotes the set of all functions $f = (f_1, \dots f_d) : U \to \mathbb{R}^d$, for which the weak partial first order derivatives $\partial_k f_i$ exist and are locally in L^d.

In order to get a more interesting class of functions, we relax the conditions in the following way:

Definition

A continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$ is called *quasiregular*, if

- $f \in W^1_{d,loc}(\mathbb{R}^d)$
- and $K_1 \geq 1$ exists, such that $||Df(x)||^d \leq K_1 J_f(x)$ a.e.,

where $W^1_{d,loc}(\mathbb{R}^d)$ denotes the set of all functions $f = (f_1, \dots f_d) : U \to \mathbb{R}^d$, for which the weak partial first order derivatives $\partial_k f_i$ exist and are locally in L^d.

Similarly there exists $K_2 \geq 1$, such that $J_f(x) \leq K_2 \ell(Df(x))^d$ a.e., where $\ell(Df(x)) := \inf_{||h||=1} ||Df(x)(h)||$.

In order to get a more interesting class of functions, we relax the conditions in the following way:

Definition

A continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$ is called *quasiregular*, if

- $f \in W^1_{d,loc}(\mathbb{R}^d)$
- and $K_1 \geq 1$ exists, such that $||Df(x)||^d \leq K_1 J_f(x)$ a.e.,

where $W^1_{d,loc}(\mathbb{R}^d)$ denotes the set of all functions $f = (f_1, \dots f_d) : U \to \mathbb{R}^d$, for which the weak partial first order derivatives $\partial_k f_i$ exist and are locally in L^d.

Similarly there exists $K_2 \geq 1$, such that $J_f(x) \leq K_2 \ell(Df(x))^d$ a.e., where $\ell(Df(x)) := \inf_{||h||=1} ||Df(x)(h)||$.

For the smallest constants K_1 and K_2 satisfying the conditions above, we call $K := \min\{K_1,\ K_2\}$ the *dilatation* of f.

In order to get a more interesting class of functions, we relax the conditions in the following way:

Definition

A continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$ is called *quasiregular*, if

- $f \in W^1_{d,loc}(\mathbb{R}^d)$
- and $K_1 \geq 1$ exists, such that $||Df(x)||^d \leq K_1 J_f(x)$ a.e.,

where $W^1_{d,loc}(\mathbb{R}^d)$ denotes the set of all functions $f = (f_1, \ldots f_d) : U \to \mathbb{R}^d$, for which the weak partial first order derivatives $\partial_k f_i$ exist and are locally in L^d.

Similarly there exists $K_2 \geq 1$, such that $J_f(x) \leq K_2 \ell(Df(x))^d$ a.e., where $\ell(Df(x)) := \inf\limits_{||h||=1} ||Df(x)(h)||$.

For the smallest constants K_1 and K_2 satisfying the conditions above, we call $K := \min\{K_1, \ K_2\}$ the *dilatation* of f.

So f maps infinitesimal balls to infinitesimal ellipsoids with bounded eccentricity.

- qr maps are open and discrete.

- qr maps are open and discrete.
- The composition of two qr maps is again qr, but in general the dilatation grows.

- qr maps are open and discrete.
- The composition of two qr maps is again qr, but in general the dilatation grows.
- There are analogues of Picard's and Montel's theorem, but for Montel's analogue we need that the iterates are uniformly qr.

- qr maps are open and discrete.
- The composition of two qr maps is again qr, but in general the dilatation grows.
- There are analogues of Picard's and Montel's theorem, but for Montel's analogue we need that the iterates are uniformly qr.
- Bergweiler and Nicks are working on an iteration theory for non-uniform qr maps by defining the Julia set of such a map by the "blowing up" property.

We want to investigate a certain qr map which can be seen as a quasiregular analogue of sine. First we start with a result by McMullen about the measure of the escaping set for a member of the sine family:

We want to investigate a certain qr map which can be seen as a quasiregular analogue of sine. First we start with a result by McMullen about the measure of the escaping set for a member of the sine family:

Theorem (McMullen 1987)

For $g(z) = \lambda \sin(z) + \mu$, $\lambda \neq 0$, the set $I(g)$ has positive area.

We want to investigate a certain qr map which can be seen as a quasiregular analogue of sine. First we start with a result by McMullen about the measure of the escaping set for a member of the sine family:

Theorem (McMullen 1987)

For $g(z) = \lambda \sin(z) + \mu,\ \lambda \neq 0$, the set $I(g)$ has positive area.

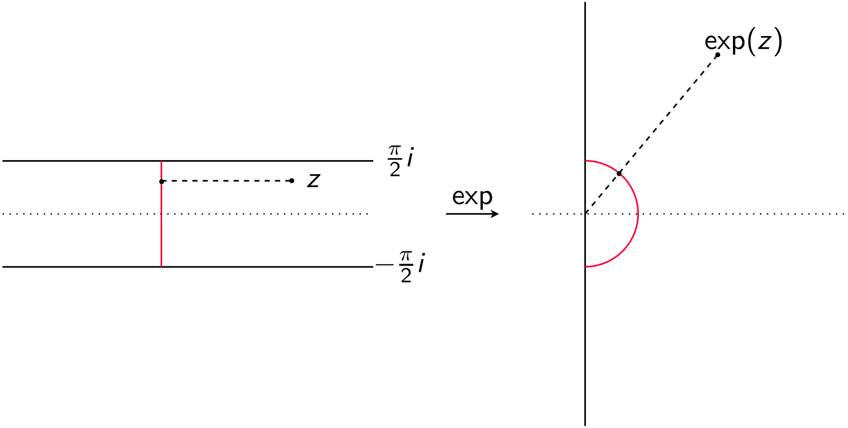
We want to generalise this theorem for a quasiregular analogue of sine.

We want to investigate a certain qr map which can be seen as a quasiregular analogue of sine. First we start with a result by McMullen about the measure of the escaping set for a member of the sine family:

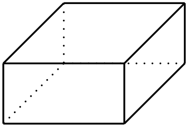
Theorem (McMullen 1987)

For $g(z) = \lambda \sin(z) + \mu$, $\lambda \neq 0$, the set $I(g)$ has positive area.

We want to generalise this theorem for a quasiregular analogue of sine. Before constructing the map, we recall the construction of the complex exponential map by using the real exponential map.

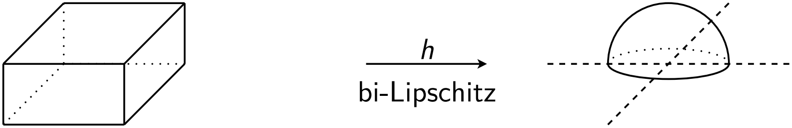

$\frac{\pi}{2}i$
z
$-\frac{\pi}{2}i$
exp
exp(z)

$\mathbb{R}^d$



$[-1, 1]^{d-1} \times [0, 1]$

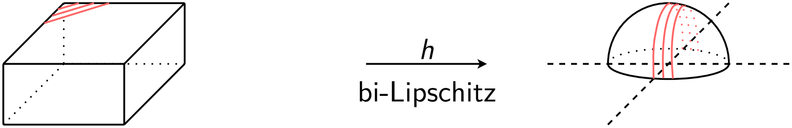
$\mathbb{R}^d$



$[-1,1]^{d-1} \times [0,1]$

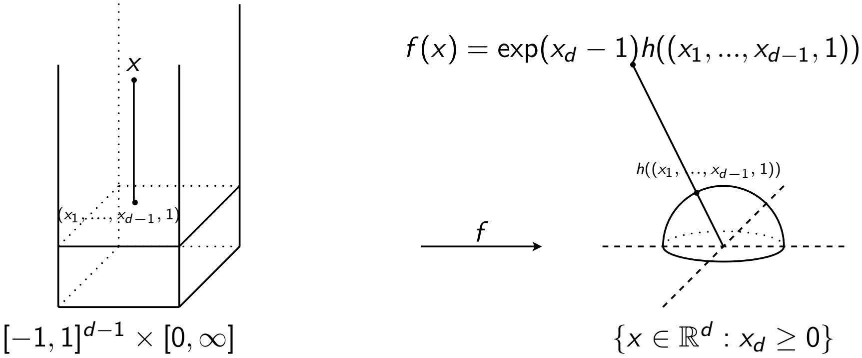
$\{x \in \mathbb{R}^d : ||x|| \leq 1,\ x_d \geq 0\}$

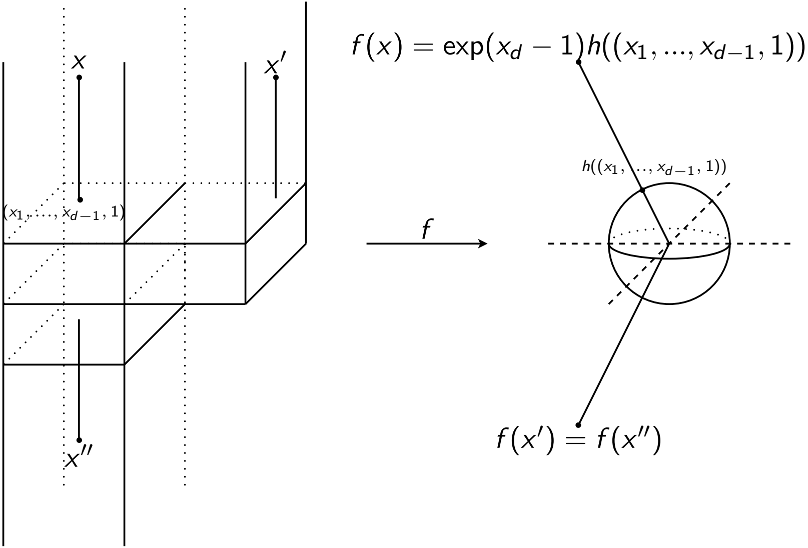
$\mathbb{R}^d$



$[-1,1]^{d-1} \times [0,1]$

$\{x \in \mathbb{R}^d : ||x|| \leq 1,\ x_d \geq 0\}$


x
$(x_1, \dots, x_{d-1}, 1)$
$[-1, 1]^{d-1} \times [0, \infty]$
$f(x) = \exp(x_d - 1)h((x_1, \dots, x_{d-1}, 1))$
f
$h((x_1, \dots, x_{d-1}, 1))$
$\{x \in \mathbb{R}^d : x_d \geq 0\}$


x
x'
$(x_1, \ldots, x_{d-1}, 1)$
x''
f
$f(x) = \exp(x_d - 1)h((x_1, \ldots, x_{d-1}, 1))$
$h((x_1, \ldots, x_{d-1}, 1))$
$f(x') = f(x'')$

- The construction of the map is due to Bergweiler and Eremenko.

- The construction of the map is due to Bergweiler and Eremenko.
- It is similar to the construction of Zorich maps, which can be seen as higher dimensional analogues of the exponential map.

- The construction of the map is due to Bergweiler and Eremenko.
- It is similar to the construction of Zorich maps, which can be seen as higher dimensional analogues of the exponential map.
- It works in any dimension $d \geq 2$.

- The construction of the map is due to Bergweiler and Eremenko.
- It is similar to the construction of Zorich maps, which can be seen as higher dimensional analogues of the exponential map.
- It works in any dimension $d \geq 2$.
- Bergweiler and Eremenko showed that the map f is in fact quasiregular.

- The construction of the map is due to Bergweiler and Eremenko.
- It is similar to the construction of Zorich maps, which can be seen as higher dimensional analogues of the exponential map.
- It works in any dimension $d \geq 2$.
- Bergweiler and Eremenko showed that the map f is in fact quasiregular.
- f is differentiable almost everywhere.

In the plane we have the following result by Schleicher:

In the plane we have the following result by Schleicher:

Theorem (Schleicher 2007)

There exists a representation of $\mathbb{C}$ *as a union of dynamic rays with the following properties: the intersection of two rays is either empty or consists of the common endpoint and the union of the rays without their endpoints has Hausdorff dimension 1.*

In the plane we have the following result by Schleicher:

Theorem (Schleicher 2007)

There exists a representation of $\mathbb{C}$ as a union of dynamic rays with the following properties: the intersection of two rays is either empty or consists of the common endpoint and the union of the rays without their endpoints has Hausdorff dimension 1.

This representation is defined by the dynamics of the map $g(z) = \lambda \sin(z) + \mu$, with parameters λ and μ chosen such that the critical values are strictly preperiodic.

In the plane we have the following result by Schleicher:

Theorem (Schleicher 2007)

There exists a representation of $\mathbb{C}$ as a union of dynamic rays with the following properties: the intersection of two rays is either empty or consists of the common endpoint and the union of the rays without their endpoints has Hausdorff dimension 1.

This representation is defined by the dynamics of the map $g(z) = \lambda \sin(z) + \mu$, with parameters λ and μ chosen such that the critical values are strictly preperiodic.
In this case $J(g) = \mathbb{C}$.

There are the following analogues of this theorem:

There are the following analogues of this theorem:

Theorem (Bergweiler and Eremenko 2011)

There exists a representation of $\mathbb{R}^d$ as a union of dynamic rays with the following properties: the intersection of two rays is either empty or consists of the common endpoint and the union of the rays without their endpoints has Hausdorff dimension 1.

There are the following analogues of this theorem:

Theorem (Bergweiler and Eremenko 2011)

There exists a representation of $\mathbb{R}^d$ as a union of dynamic rays with the following properties: the intersection of two rays is either empty or consists of the common endpoint and the union of the rays without their endpoints has Hausdorff dimension 1.

This representation is defined by the dynamics of the (locally expanding) map λf, λ sufficiently large.

There are the following analogues of this theorem:

Theorem (Bergweiler and Eremenko 2011)

There exists a representation of $\mathbb{R}^d$ as a union of dynamic rays with the following properties: the intersection of two rays is either empty or consists of the common endpoint and the union of the rays without their endpoints has Hausdorff dimension 1.

This representation is defined by the dynamics of the (locally expanding) map λf, λ sufficiently large.

Theorem (Fletcher and Nicks 2012)

For λ sufficiently large, the periodic points of $\widetilde{f} = \lambda f$ are dense in $\mathbb{R}^d$ (and all repelling).

There are the following analogues of this theorem:

Theorem (Bergweiler and Eremenko 2011)

There exists a representation of $\mathbb{R}^d$ as a union of dynamic rays with the following properties: the intersection of two rays is either empty or consists of the common endpoint and the union of the rays without their endpoints has Hausdorff dimension 1.

This representation is defined by the dynamics of the (locally expanding) map λf, λ sufficiently large.

Theorem (Fletcher and Nicks 2012)

For λ sufficiently large, the periodic points of $\widetilde{f} = \lambda f$ are dense in $\mathbb{R}^d$ (and all repelling).
Furthermore $\widetilde{f}$ has the blowing-up property everywhere in $\mathbb{R}^d$, that is

$$\bigcup_{k=0}^{\infty} \widetilde{f}^k(U) = \mathbb{R}^d, \quad \text{for any non-empty open set } U \subset \mathbb{R}^d.$$

In the case where the map f is not chosen to be locally expanding, we have the following analogue of McMullen's theorem:

In the case where the map f is not chosen to be locally expanding, we have the following analogue of McMullen's theorem:

Theorem

Let f be the quasiregular analogue of sine. Then

$$\text{meas}(I(f)) > 0,$$

where meas *denotes the d-dimensional Lebesgue measure.*

The proof uses some ideas of Aspenberg and Bergweiler (2010).

The proof uses some ideas of Aspenberg and Bergweiler (2010).
We are going to show, that the set of points which escape exponentially fast in direction x_d has positive measure.

The proof uses some ideas of Aspenberg and Bergweiler (2010).
We are going to show, that the set of points which escape exponentially fast in direction x_d has positive measure.
Let

$$L := \left\{ x \in \mathbb{R}^d : |f_d(x)| \geq \exp\left(\frac{1}{2}|x_d|\right) \right\}$$

The proof uses some ideas of Aspenberg and Bergweiler (2010).
We are going to show, that the set of points which escape exponentially fast in direction x_d has positive measure.
Let

$$L := \left\{ x \in \mathbb{R}^d : |f_d(x)| \geq \exp\left(\frac{1}{2}|x_d|\right) \right\}$$

and

$$T := \{x \in L : f^n(x) \in L \text{ for all } n \in \mathbb{N}\}.$$

The proof uses some ideas of Aspenberg and Bergweiler (2010).
We are going to show, that the set of points which escape exponentially fast in direction x_d has positive measure.
Let

$$L := \left\{ x \in \mathbb{R}^d : |f_d(x)| \geq \exp\left(\frac{1}{2}|x_d|\right) \right\}$$

and

$$T := \{x \in L : f^n(x) \in L \text{ for all } n \in \mathbb{N}\}.$$

Then $T \subset I(f)$. For $n \geq 0$ let

$$T_n := \{x \in L : f^k(x) \in L \text{ for } 0 \leq k \leq n\}.$$

The proof uses some ideas of Aspenberg and Bergweiler (2010).
We are going to show, that the set of points which escape exponentially fast in direction x_d has positive measure.
Let

$$L := \left\{ x \in \mathbb{R}^d : |f_d(x)| \geq \exp\left(\frac{1}{2}|x_d|\right) \right\}$$

and

$$T := \{x \in L : f^n(x) \in L \text{ for all } n \in \mathbb{N}\}.$$

Then $T \subset I(f)$. For $n \geq 0$ let

$$T_n := \{x \in L : f^k(x) \in L \text{ for } 0 \leq k \leq n\}.$$

Then $L = T_0$ and we put

$$S := \mathbb{R}^d \setminus L.$$

Definition

For $A, B \subset \mathbb{R}^d$ measurable we denote the density of A in B by

$$\text{dens}(A, B) := \frac{\text{meas}(A \cap B)}{\text{meas}(B)}.$$

Definition

For $A, B \subset \mathbb{R}^d$ measurable we denote the density of A in B by

$$\operatorname{dens}(A, B) := \frac{\operatorname{meas}(A \cap B)}{\operatorname{meas}(B)}.$$

For $x \in \mathbb{R}^d$ we denote by

$$Q(x) := \left\{ y \in \mathbb{R}^d : |y_j - x_j| \leq \frac{|x_d|}{2} \right\}$$

the axis parallel cube around x with edges of length $\frac{|x_d|}{2}$.

Lemma

For x_0 large and $|x_d| \geq x_0$ we have

$$\operatorname{dens}(S, Q(x)) \leq 2\widetilde{L}^4 \exp\left(-\frac{|x_d|}{4} + \frac{1}{2}\right) =: 2\widetilde{L}^4 \delta(|x_d|),$$

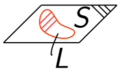
where $\widetilde{L}$ denotes the Lipschitz constant of $f|_{[-1,1]^{d-1}\times[0,1]}$.

Lemma

For x_0 large and $|x_d| \geq x_0$ we have

$$\operatorname{dens}(S, Q(x)) \leq 2\widetilde{L}^4 \exp\left(-\frac{|x_d|}{4} + \frac{1}{2}\right) =: 2\widetilde{L}^4 \delta(|x_d|),$$

where $\widetilde{L}$ denotes the Lipschitz constant of $f|_{[-1,1]^{d-1}\times[0,1]}$.

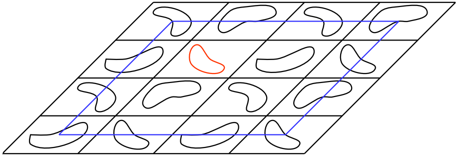


Lemma

For x_0 large and $|x_d| \geq x_0$ we have

$$\text{dens}(S, Q(x)) \leq 2\widetilde{L}^4 \exp\left(-\frac{|x_d|}{4} + \frac{1}{2}\right) =: 2\widetilde{L}^4 \delta(|x_d|),$$

where $\widetilde{L}$ denotes the Lipschitz constant of $f|_{[-1,1]^{d-1}\times[0,1]}$.

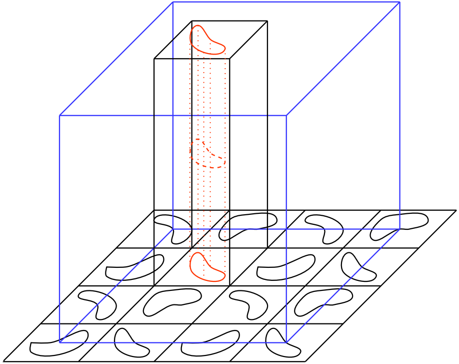


Lemma

For x_0 large and $|x_d| \geq x_0$ we have

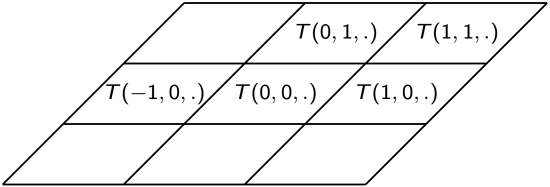
$$\text{dens}(S, Q(x)) \leq 2\widetilde{L}^4 \exp\left(-\frac{|x_d|}{4} + \frac{1}{2}\right) =: 2\widetilde{L}^4 \delta(|x_d|),$$

where $\widetilde{L}$ denotes the Lipschitz constant of $f|_{[-1,1]^{d-1}\times[0,1]}$.



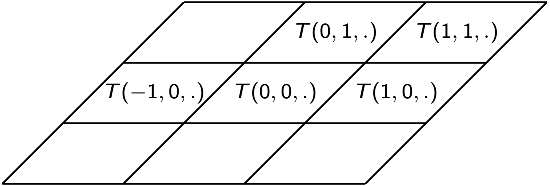
We put $R := \mathbb{Z}^{d-1} \times \{-1, 1\}$ and for $r = (r_1, ..., r_d) \in R$ we define

$$T(r) := \{x \in \mathbb{R}^d : |x_j - 2r_j| \leq 1 \text{ for } 1 \leq j \leq d-1,\ r_d x_d \geq 0\}$$



We put $R := \mathbb{Z}^{d-1} \times \{-1, 1\}$ and for $r = (r_1, ..., r_d) \in R$ we define

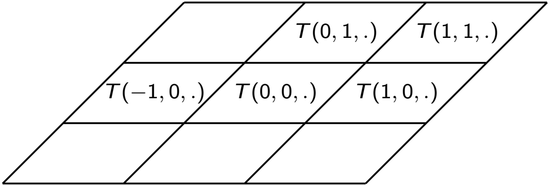
$$T(r) := \{x \in \mathbb{R}^d : |x_j - 2r_j| \leq 1 \text{ for } 1 \leq j \leq d-1,\ r_d x_d \geq 0\}$$



Depending on r, f maps $T(r)$ bijectively onto $\{x \in \mathbb{R}^d : x_d \geq 0\}$ or onto $\{x \in \mathbb{R}^d : x_d \leq 0\}$.

We put $R := \mathbb{Z}^{d-1} \times \{-1, 1\}$ and for $r = (r_1, ..., r_d) \in R$ we define

$$T(r) := \{x \in \mathbb{R}^d : |x_j - 2r_j| \leq 1 \text{ for } 1 \leq j \leq d-1,\ r_d x_d \geq 0\}$$



Depending on r, f maps $T(r)$ bijectively onto $\{x \in \mathbb{R}^d : x_d \geq 0\}$ or onto $\{x \in \mathbb{R}^d : x_d \leq 0\}$.
For $r \in R$ we denote by Λ^r the inverse function of $f|_{T(r)}$, thus $\Lambda^r : \mathbb{H}^+ \to T(r)$ or $\Lambda^r : \mathbb{H}^- \to T(r)$ depending on r.

Let $u \in T_{n-1}$ with $|u_d| > x_0$.

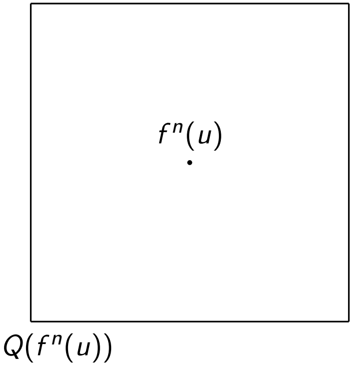
Let $u \in T_{n-1}$ with $|u_d| > x_0$. Denote by $\underline{s}$ the external address of u, i.e. the sequence $\underline{s} = (s_k)_{k\geq 0}$ in R with $f^k(x) \in T(s_k)$

Let $u \in T_{n-1}$ with $|u_d| > x_0$. Denote by $\underline{s}$ the external address of u, i.e. the sequence $\underline{s} = (s_k)_{k\geq 0}$ in R with $f^k(x) \in T(s_k)$ and put

$$\varphi_n := \Lambda^{s_0} \circ ... \circ \Lambda^{s_{n-1}}.$$

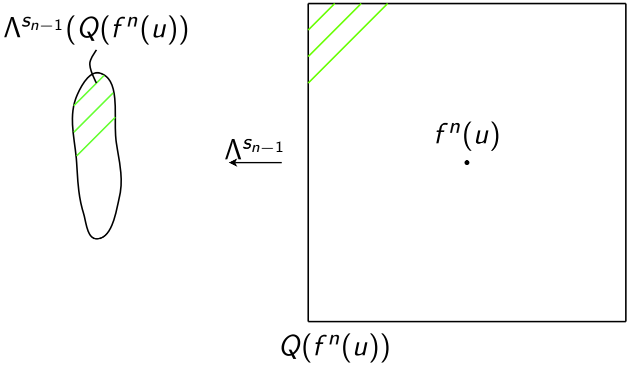
Let $u \in T_{n-1}$ with $|u_d| > x_0$. Denote by $\underline{s}$ the external address of u, i.e. the sequence $\underline{s} = (s_k)_{k \geq 0}$ in R with $f^k(x) \in T(s_k)$ and put

$$\varphi_n := \Lambda^{s_0} \circ \ldots \circ \Lambda^{s_{n-1}}.$$



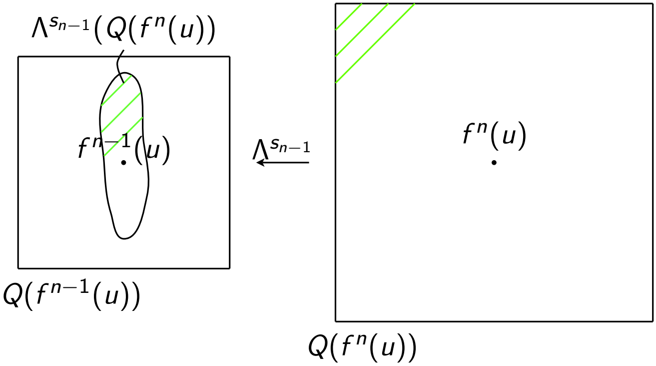
Let $u \in T_{n-1}$ with $|u_d| > x_0$. Denote by $\underline{s}$ the external address of u, i.e. the sequence $\underline{s} = (s_k)_{k \geq 0}$ in R with $f^k(x) \in T(s_k)$ and put

$$\varphi_n := \Lambda^{s_0} \circ ... \circ \Lambda^{s_{n-1}}.$$



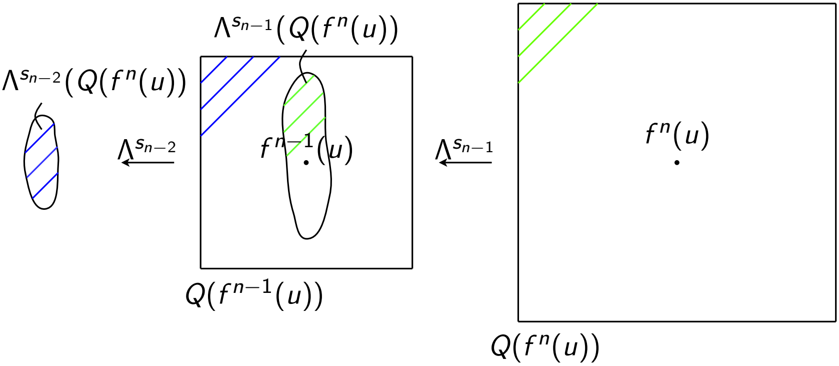
Let $u \in T_{n-1}$ with $|u_d| > x_0$. Denote by $\underline{s}$ the external address of u, i.e. the sequence $\underline{s} = (s_k)_{k \geq 0}$ in R with $f^k(x) \in T(s_k)$ and put

$$\varphi_n := \Lambda^{s_0} \circ \ldots \circ \Lambda^{s_{n-1}}.$$



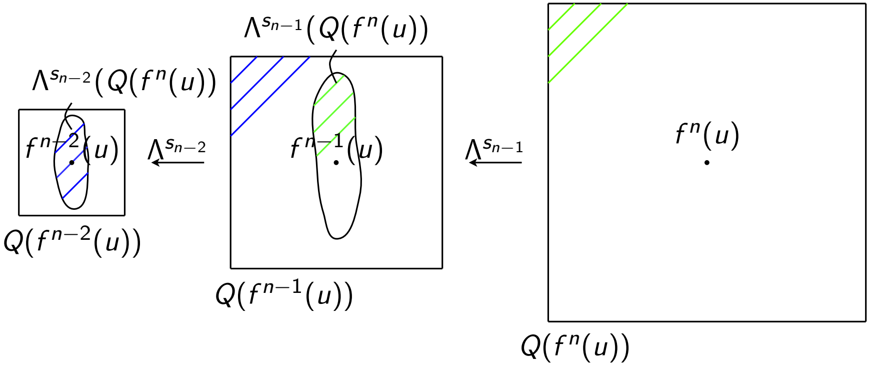
Let $u \in T_{n-1}$ with $|u_d| > x_0$. Denote by $\underline{s}$ the external address of u, i.e. the sequence $\underline{s} = (s_k)_{k \geq 0}$ in R with $f^k(x) \in T(s_k)$ and put

$$\varphi_n := \Lambda^{s_0} \circ \ldots \circ \Lambda^{s_{n-1}}.$$



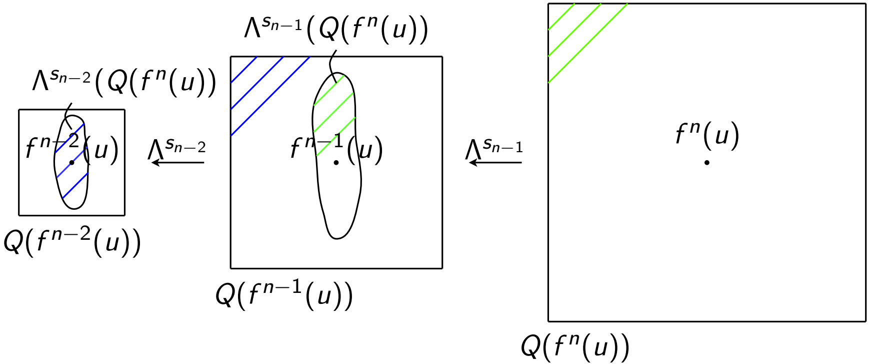
Let $u \in T_{n-1}$ with $|u_d| > x_0$. Denote by $\underline{s}$ the external address of u, i.e. the sequence $\underline{s} = (s_k)_{k\geq 0}$ in R with $f^k(x) \in T(s_k)$ and put

$$\varphi_n := \Lambda^{s_0} \circ \ldots \circ \Lambda^{s_{n-1}}.$$



Let $u \in T_{n-1}$ with $|u_d| > x_0$. Denote by $\underline{s}$ the external address of u, i.e. the sequence $\underline{s} = (s_k)_{k \geq 0}$ in R with $f^k(x) \in T(s_k)$ and put

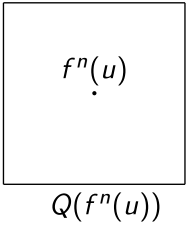
$$\varphi_n := \Lambda^{s_0} \circ \ldots \circ \Lambda^{s_{n-1}}.$$



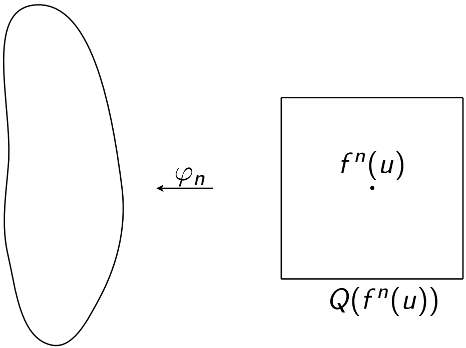
So we have $\Lambda^{s_{n-j-1}}(Q(f^{n-j}(u))) \subset Q(f^{n-j-1}(u))$ for x_0 large and $0 \leq j \leq n-1$.

We denote by $B(u, r_n(u)) = B(u, r_n)$ the largest ball around u in $\varphi_n(Q(f^n(u)))$.

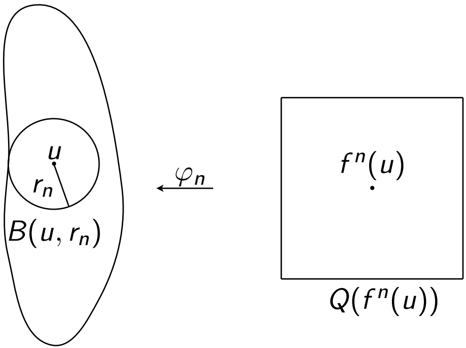
We denote by $B(u, r_n(u)) = B(u, r_n)$ the largest ball around u in $\varphi_n(Q(f^n(u)))$.



We denote by $B(u, r_n(u)) = B(u, r_n)$ the largest ball around u in $\varphi_n(Q(f^n(u)))$.



We denote by $B(u, r_n(u)) = B(u, r_n)$ the largest ball around u in $\varphi_n(Q(f^n(u)))$.

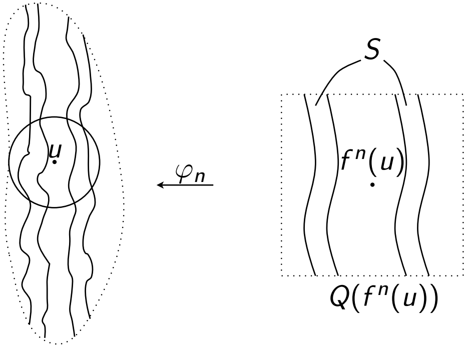


Lemma

For x_0 large we have

$$\operatorname{dens}(f^{-n}(S), B(u, r_n)) \leq \tilde{\eta}\delta\left(\mathrm{E}^n_{\frac{1}{2}}(x_0)\right) K^n =: \tilde{\eta}\delta(x_{0,n})K^n$$

for some constants $\tilde{\eta}, K \geq 1$, where $E_{\frac{1}{2}} : \mathbb{R} \to \mathbb{R},\ x \mapsto \exp\left(\frac{1}{2}x\right)$



$$\text{dens}(f^{-n}(S), B(u, r_n))$$

$$\begin{aligned}&\operatorname{dens}(f^{-n}(S), B(u, r_n)) \\ &\leq \frac{2^d}{c_d} \operatorname{dens}(S, Q(f^n(u))) \frac{\sup_{y \in Q(f^n(u))} |J_{\varphi_n}(y)|}{\left(\inf_{y \in Q(f^n(u))} \ell(D\varphi_n(y)) \right)^d}\end{aligned}$$

$$\begin{aligned}
&\operatorname{dens}(f^{-n}(S), B(u, r_n)) \\
&\leq \frac{2^d}{c_d} \operatorname{dens}(S, Q(f^n(u))) \frac{\sup_{y \in Q(f^n(u))} |J_{\varphi_n}(y)|}{\left(\inf_{y \in Q(f^n(u))} \ell(D\varphi_n(y)) \right)^d} \\
&\leq \frac{2^d}{c_d} \delta \left(\mathrm{E}^n_{\frac{1}{2}}(x_0) \right) \prod_{j=0}^{n-1} \frac{\sup_{y \in Q(f^{n-j}(u))} |J_{\Lambda^{s_{n-j-1}}}(y)|}{\left(\inf_{y \in Q(f^{n-j}(u))} \ell(D_{\Lambda^{s_{n-j-1}}}(y)) \right)^d}
\end{aligned}$$

$$\begin{aligned}
&\operatorname{dens}(f^{-n}(S), B(u, r_n)) \\
&\leq \frac{2^d}{c_d} \operatorname{dens}(S, Q(f^n(u))) \frac{\sup\limits_{y \in Q(f^n(u))} |J_{\varphi_n}(y)|}{\left(\inf\limits_{y \in Q(f^n(u))} \ell(D\varphi_n(y)) \right)^d} \\
&\leq \frac{2^d}{c_d} \delta \left(\mathrm{E}^n_{\frac{1}{2}}(x_0) \right) \prod_{j=0}^{n-1} \frac{\sup\limits_{y \in Q(f^{n-j}(u))} |J_{\Lambda^{s_{n-j-1}}}(y)|}{\left(\inf\limits_{y \in Q(f^{n-j}(u))} \ell(D_{\Lambda^{s_{n-j-1}}}(y)) \right)^d} \\
&\leq \frac{2^d}{c_d} \delta \left(\mathrm{E}^n_{\frac{1}{2}}(x_0) \right) \left(\frac{c_3}{c_1^d} \left(1 + 2\sqrt{d} \right)^d \right)^n
\end{aligned}$$

Theorem (Besicovitch covering lemma)

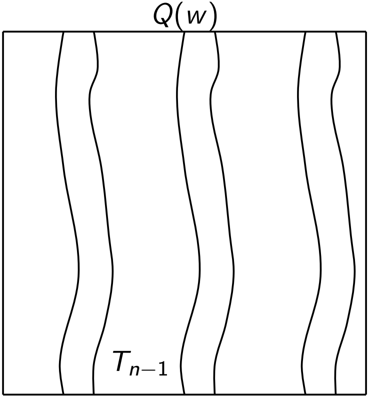
Let $M \subset \mathbb{R}^d$ be bounded, $r : M \to]0, \infty[$. Then there exists an at most countable subset A of M satisfying

$$M \subset \bigcup_{x \in A} B(x, r(x))$$

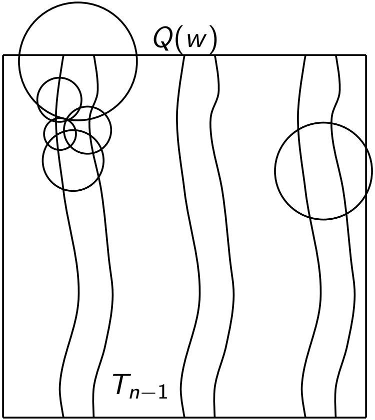
such that no point in $\mathbb{R}^d$ is contained in more than 4^{2d} of the balls $B(x, r(x))$, $x \in A$.

Now let $w \in \mathbb{R}^d$ with $|w_d| > 2x_0$.

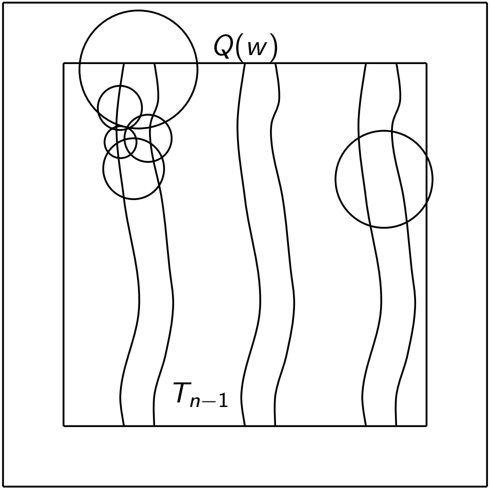
Now let $w \in \mathbb{R}^d$ with $|w_d| > 2x_0$.



Now let $w \in \mathbb{R}^d$ with $|w_d| > 2x_0$.



Now let $w \in \mathbb{R}^d$ with $|w_d| > 2x_0$.



So we get the following

Lemma

$$\text{dens}(f^{-n}(S), T_{n-1} \cap Q(w)) = \text{dens}(T_{n-1} \setminus T_n, T_{n-1} \cap Q(w)) \leq \eta\delta(x_{0,n})K^n$$

So we get the following

Lemma

$$\operatorname{dens}(f^{-n}(S), T_{n-1} \cap Q(w)) = \operatorname{dens}(T_{n-1} \setminus T_n, T_{n-1} \cap Q(w)) \leq \eta\delta(x_{0,n})K^n$$

Lemma

For x_0 large, the product

$$\prod_{k=1}^{\infty} \left(1 - \eta\delta(x_{0,n})K^k\right)$$

converges and we have

So we get the following

Lemma

$$\mathrm{dens}(f^{-n}(S), T_{n-1} \cap Q(w)) = \mathrm{dens}(T_{n-1} \setminus T_n, T_{n-1} \cap Q(w)) \leq \eta\delta(x_{0,n})K^n$$

Lemma

For x_0 large, the product

$$\prod_{k=1}^{\infty} \left(1 - \eta\delta(x_{0,n})K^k\right)$$

converges and we have

$$\mathrm{dens}(T_n, T_0 \cap Q(w)) \geq \prod_{k=1}^{\infty} \left(1 - \eta\delta(x_{0,n})K^k\right) > 0.$$

Hence we have

$$\text{dens}(T, T_0 \cap Q(w)) > 0$$

and $\text{meas}(T) > 0$ and since $T \subset I(f)$ we get $\text{meas}(I(f)) > 0$.

By using the technique of the proof, we get a similar result as Schubert:

By using the technique of the proof, we get a similar result as Schubert:

Theorem (Schubert 2008)

Let S be a strip of width 2π. Then $S \setminus I(\sinh)$ has finite area.

By using the technique of the proof, we get a similar result as Schubert:

Theorem (Schubert 2008)

Let S be a strip of width 2π. Then $S \setminus I(\sinh)$ has finite area.

In the case of the quasiregular analogue of sine we have

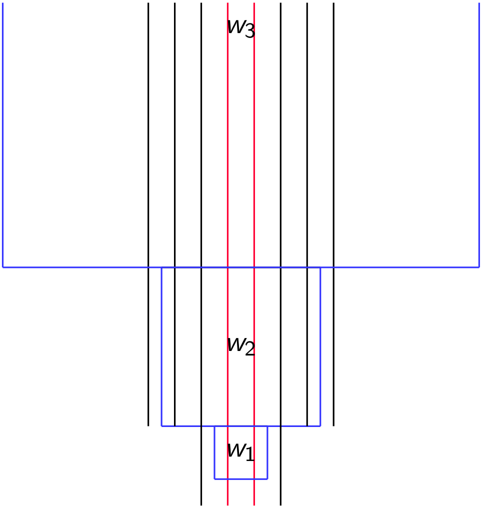
Theorem

Let $T(r)$ be a tract of f. Then $T(r) \setminus I(f)$ has finite measure.

During the proof of the first theorem we showed, that

$$\operatorname{dens}(T, T_0 \cap Q(w)) \geq \prod_{k=1}^{\infty} \left(1 - \eta\delta(x_{0,n}) K^k\right)$$

for all w with $|w_d| > 2x_0$, for large x_0.
Now we cover the initial tract $T((0, ..., 0, 1))$ with cubes $Q(w_j)$ in the following way:


W_3
W_2
W_1

Thank you very much for your attention.