Omitted Values and Herman rings

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TOPICS IN COMPLEX DYNAMICS

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Presentation Outline

The Setting

Let $f \; : \; \mathbb{C} \to \widehat{\mathbb{C}}$ be transcendental meromorphic map

- A single essential singularity - at ∞.
- f can have asymptotic values along with critical values.
- $O^{-}(\infty)$ finite means f is entire/analytic self map of $\mathbb{C}^*$.
- $O^{-}(\infty)$ can be an infinite set. f can have finitely many poles. The case of our interest.

The Herman ring

Definition (Fatou set)

$\mathcal{F}(f) =$
$\{z \ : \ \{f^n\}_{n>0}$ is defined & normal in a neighborhood of $z\}$

A periodic Fatou component can be one of the five types:

- Attracting domain
- Parabolic domain
- Baker domain
- Siegel disk
- Herman ring, doubly connected by definition

More on Herman rings

- Doubly connected by definition.
- Consists of f^p-invariant CURVES.
- Never completely invariant under any f^k.
- One Herman ring implies infinitely many multiply connected Fatou components.

Omitted values

- A special kind of singular value.
- Each singularity lying over it, is direct.
- Probably the simplest instance of transc. singularity.
- At most two in number.
- Cannot be in a Herman ring.

Presentation Outline

Known facts on Herman rings

1. Polynomials do not have Herman rings. But rationals can have.
2. Transc. entire maps do not have them.
3. Analytic self maps of $\mathbb{C}^*$ (with single essential singularity) can have them but at most one and of period one.
4. Transc. (general) meromorphic functions are known to have Herman rings.

Observation

Points with finite/empty backward orbit

- Always exists in cases $(1), (3)$ and (4).
- They seem to control not only Herman rings but MANY other aspects of dynamics.

Motivation: Directions of generalisations

1. Rational functions
2. Finite/Bounded type maps; control on singular values
3. Transc. mero. with finitely many poles;control on poles
4. Transc. entire functions; $O^{-}(\infty) = \emptyset$
5. Analytic self maps of $\mathbb{C}^*$;$O^{-}(\infty)$ is finite

   ....

6. General transc. mero. (at least two poles or one pole which is not omitted) with at least one omitted value.These maps
   - can be of unbounded type
   - can have infinitely many poles

[6] can be viewed as a generalisation of [4] and [5].

Motivation

Assumption

f is general meromorphic and has at least one omitted value.

Theorem (N-Jheng, 2011)

U- Fatou component, $c(U) > 1 \Rightarrow \exists n$ such that U_n

- contains all omitted values; Examples known.
- is an infinitely connected Baker domain with period > 1.
- is a wandering domain; Buried Julia components exist.
- is a Herman ring.

Conjecture for gen. mero. maps

Each doubly periodic Fatou component is a Herman ring.

True when there is an omitted value.

Presentation Outline

Can a Herman ring really occur?

γ is a closed curve non-contractible in the Fatou set

1. $\exists n$ and $\gamma_n \subseteq f^n(\gamma)$ such that $B(\gamma_n) \ni$ a pole.
2. $O_f \subset B(\gamma_{n+1})$ for some $\gamma_{n+1} \subset f(\gamma_n)$.
3. $f : \gamma \to \mathbb{C}$ is one-one $\Rightarrow f : B(\gamma) \to \widehat{\mathbb{C}}$ is one-one.

Can a Herman ring really occur?

Theorem

1. *Each Julia component containing pole/at least one omitted value is unbounded* $\Rightarrow$ *No Herman ring.*
2. *All the poles are multiple* $\Rightarrow$ *No Herman ring.*

Proof.

1. By (1) and (2).
2. By (3).

□

Arrangement of Herman rings is Restricted!

Theorem

No nested or strictly non-nested Herman rings exist.

Proof.

Let $f^p(H) = H$, $\gamma \subset H$ is a Jordan curve.
$f^p(\gamma) = \gamma$ and non-contractible in H.

- Nested:
  $\exists \gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}$ such that $\gamma_{n_i} \subset H_{n_i}$ and H_{n_i} is an innermost or the outermost.
  A is the region bounded by $\{\gamma_{n_1}, \gamma_{n_2}, ..., \gamma_{n_k}\}$.
  $f : A \to \mathbb{C}$ analytic and $f(A) = A$: Contradiction.
- Strictly Non-Nested:
  $\exists$ at least two maximal nests enclosing poles.
  The maximal nest enclosing O_f has at least two rings: Contradiction.

Arrangement of Herman rings...contd

Corollary

- No Herman ring with period 1 or 2.
- At least two poles but one pole is omitted $\Rightarrow$ No Herman ring.

Proof.

One periodic Herman ring is nested.
Two periodic Herman ring is nested or strictly non-nested.
One omitted pole means nested Herman ring. $\square$

Maps with single pole

Theorem

Single pole $\Rightarrow$ No Herman ring.

Proof.

H - p-periodic Herman ring.

- $\exists H_l$ enclosing the pole w.
- Let all of them be $H_{i_1}, H_{i_2}, ..., H_{i_n}$; H_{i_1} be innermost and H_{i_n} be the outermost w.r.t. w.

Lemma

$B(H_{n_k}) \bigcap O_f = \emptyset$ and no pre-image of w is in $B(H_{n_k})$.

- $\{H_{i_1+1}, H_{i_2+1}, ..., H_{i_n+1}\}$ are in a maximal nest other than that containing H_{n_i}s. H_{i_n+1} is the innermost.
- $\exists m$ (consider the least)st H_{i_n+1+m} encloses w.

□

Proof...contd.

- $H_{i_n+1+m} = H_{i_1}$ as w is the only pole (Otherwise there will be at least $n+1$ rings enclosing w).
- But $H_{i_1+1+m} = H_{i_n}$ and $H_{i_1+1+m+1+m} = H_{i_1}$. That means $p = 2(m+1)$.
- Take a Jordan curve $\gamma_1 \subset H_{i_1}$. Then $f^{m+1}(\gamma_1) \subset H_{i_n}$ is Jordan.
- Consider the annulus A bounded by these two curves.
- $f^{m+1}(A) = A$ and $f : A \to A$ analytic: Contradiction.

□

Example: $\frac{e^{g(z)}}{(z-z_0)^m}$ has no Herman ring.

Similar arguments can also prove that

Theorem

If there are only two maximal nests, one is strictly nested and other has only one ring then the Herman ring is odd-periodic.

Presentation Outline

The question remains.....

$\exists$ a mero. map with an omitted value and a Herman ring **?**

Thank you.