Combinatorics and Topology of the Multicorns

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Basic Definitions

We consider the iteration of unicritical antiholomorphic polynomials $\mathbf{f}_{d,c} = \bar{z}^d + c$ for any degree $d \geq 2$ and $c \in \mathbb{C}$. In analogy to the holomorphic case, we define the Julia, Fatou and filled-in Julia set of $\mathbf{f}_{d,c}$ as:

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Definition

The **Multicorn** of degree d is defined as $\mathcal{M}_d^* = \{c \in \mathbb{C} : \mathbf{K}(\mathbf{f}_{d,c})$ is connected $\}$

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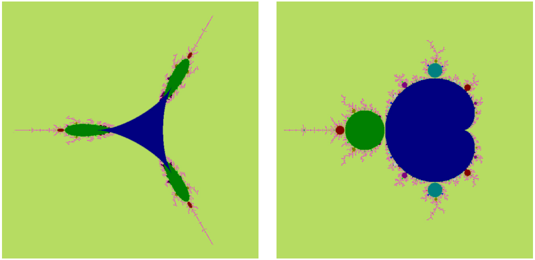

Left: The Tricorn ($\mathcal{M}_2^*$) . Right: The Mandelbrot set ($\mathcal{M}_2$).

Earlier work

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Theorem (Nakane)

The map $\Phi : \mathbf{C} \setminus \mathcal{M}_d^* \to \mathbf{C} \setminus \overline{\mathbb{D}}$, defined by $c \mapsto \phi_c(c)$ (where ϕ_c is the Bottcher coordinate near ∞) is a real-analytic diffeomorphism. In particular, the Multicorns are connected.

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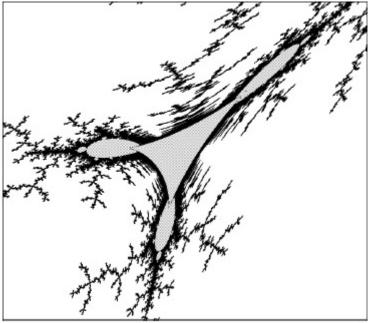
The previous theorem also allows us to define parameter rays of the Multicorns as pre-images of radial lines in $\mathbf{C} \setminus \overline{\mathbf{D}}$ under the map Φ. It is worth noting that the parameter dependence of the Bottcher coordinate is only real-analytic in this case.

Nakane and Schleicher investigated hyperbolic components of the Multicorns [NS1] and gave natural parametrizations for them. Milnor, in a seminal paper [Mi1], investigated real cubic polynomials and identified antiholomorphic quadratic polynomials as a prototypical real form. One of Milnor's conjectures was that the tricorn is not pathwise connected. This conjecture was established in recent work by Hubbard and Schleicher [HS].



An apparently embedded tricorn in the space of real cubic polynomials from Milnor's study [Mi1].

Orbit Portraits

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Let $\mathcal{O} = \{z_1, z_2, \cdots, z_k\}$ be a periodic cycle of a unicritical antipolynomial **f**. If a dynamic ray $\mathcal{R}_t^f$ at a rational angle t lands at some z_i; then for all j, the set $\mathcal{A}_j$ of the angles of all the dynamic rays landing at z_j is a non-empty finite subset of $\mathbb{Q}/\mathbb{Z}$. The collection $\{\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_k\}$ will be called the **Orbit Portrait** $\mathcal{P}(\mathcal{O})$ of the orbit $\mathcal{O}$.

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For any antipolynomial $\mathbf{f}$, if the external ray $\mathcal{R}_t$ at angle t lands at a point $z \in \mathcal{J}(f)$, then the image ray $\mathbf{f}(\mathcal{R}_t) = \mathcal{R}_{-dt}$ lands at the point $\mathbf{f}(z)$. Furthermore, if three or more external rays land at z, then the cyclic order of their angles around $\mathbf{R}/\mathbf{Z}$ is reversed by the action of $\mathbf{f}$.

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Lemma (Structure of orbit portraits)

Let $\mathcal{P}(\mathcal{O}) = \{\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_k\}$ *be the (non-trivial) orbit portrait associated with an orbit of period* k *of a unicritical antipolynomial* $\mathbf{f}$*. Then each* $\theta \in \mathcal{A}_j$ *is periodic under* $\theta \to -d\theta$ *and there are four possibilities for their periods:*

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- *If k is even, then all angles in $\mathcal{P}(\mathcal{O})$ have the same period rk for some $r \geq 1$.*

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- *$|\mathcal{A}_j| = 2$ and both angles have period k.*
- *$|\mathcal{A}_j| = 2$ and both angles have period $2k$.*
- *$|\mathcal{A}_j| = 3$; one angle has period k and the other two angles have period $2k$. One of the characteristic angles has exact period k and the other has exact period $2k$.*

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- *$|\mathcal{A}_j| = 3$; one angle has period k and the other two angles have period $2k$. One of the characteristic angles has exact period k and the other has exact period $2k$.*

All the above possibilities are realized.

Bifurcation phenomenon and q.c. conjugacy

Theorem (Nakane, Schleicher)

The boundary of a hyperbolic component of odd period k consists entirely of parameters having a parabolic orbit of exact period k. In local conformal coordinates, the 2k-th iterate of such a map has the form $z \to z + z^{q+1} + \cdots$ with $q \in \{1, 2\}$.

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A parameter c will be called a **parabolic cusp** point if it has a parabolic periodic point of odd period such that $q = 2$ in the previous theorem. It turns out that there are only finitely many cusp points of a given (odd) period.

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Theorem (Nakane, Schleicher)

Every non-cusp parabolic parameter lies in the interior of a real-analytic arc consisting of non-cusp parabolic parameters with quasiconformally equivalent but conformally inequivalent dynamics. These arcs are called **parabolic arcs**. *Further, each parabolic arc has two cusp points at its two ends.*

Bifurcation phenomenon and q.c. conjugacy

Definition (Root and co-root arcs)

We call a parabolic arc a **root arc** if, in the dynamics of any parameter on this arc, the parabolic orbit disconnects the Julia set. Otherwise, we call it a co-root arc.

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Consider a parabolic arc of a multicorn for which the parabolic orbits have odd period k. Then at both ends there are subarcs along which a bifurcation from period k to period $2k$ occurs.

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Theorem (Nakane, Schleicher)

For every even period k and every multiplier μ with $|\mu| \leq 1$, the set of parameters $c \in \mathbb{C}$ for which $\bar{z}^d + c$ has a periodic orbit with exact period k and multiplier μ is finite.

Periodic parameter rays and structure of hyperbolic components

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Let k be any odd integer. Then,

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- *Every rational parameter ray of period $4k$ lands at a parabolic parameter of ray period $4k$.*
- *The boundary of every hyperbolic component of odd period $k(\neq 1)$ consists of exactly $d+1$ parabolic arcs and the same number of cusp points. d of these are co-root parabolic arcs and on each of them, the parabolic orbit portrait is trivial and constant. Exactly one parameter ray of period k accumulates there.*

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- *On the remaining (root) parabolic arc, the parabolic orbit portrait is constant and non-trivial. This arc contains the set of accumulation points of two periodic parameter rays of period $2k$.*

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- *The boundary of every hyperbolic component of period* $2k$ *(twice an odd integer) which bifurcates from a hyperbolic component of period* k *contains exactly* $d - 2$ *co-roots (landing point of a single periodic parameter ray of period* $2k$*) and no root (landing point of exactly two parameter rays of period* $2k$*).*

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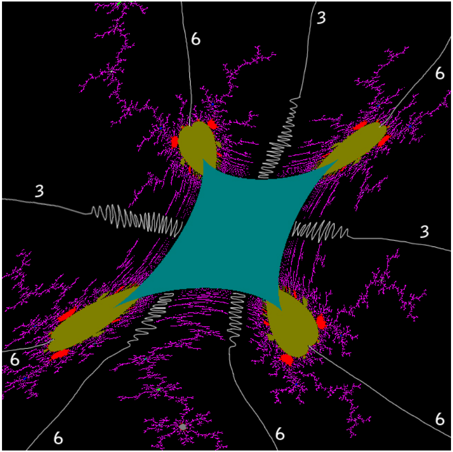
- *The boundary of every hyperbolic component of period $2k$ (twice an odd integer) which bifurcates from a hyperbolic component of period k contains exactly $d-2$ co-roots (landing point of a single periodic parameter ray of period $2k$) and no root (landing point of exactly two parameter rays of period $2k$).*
- *The boundary of every other hyperbolic component contains exactly $d-2$ co-roots and one root.*

Pictorial illustration



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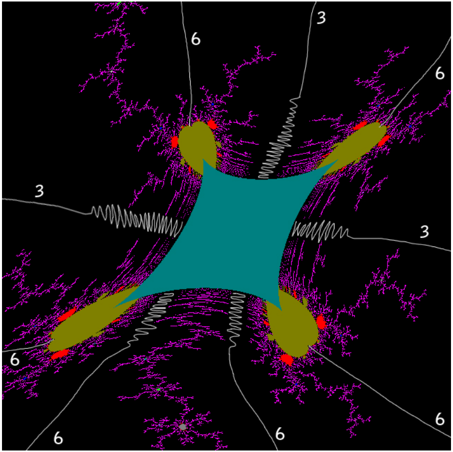

Figure: Zoom of $\mathcal{M}_3^*$ near a hyperbolic component of period 3 with the bifurcated period 6 components. The ray landing/accumulation patterns are shown.

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- Hyperbolic components of even period which do not bifurcate from any odd period hyperbolic component.

The analysis in the last case is similar to that in the holomorphic case and can be found in [Eberlein, Mukherjee, Schleicher]. Here and in [Mukherjee, Nakane, Schleicher], we emphasize on the first two cases, which are the specialities of the anti-holomorphic parameter spaces.

Dynamic roots and co-roots

Definition (Roots and Co-Roots of Fatou Components)

Let z be a boundary point of a periodic Fatou component U corresponding to a (super-)attracting or parabolic unicritical anti-polynomial so that the first return map of U fixes z. Then we call z a **root** of U if it disconnects the filled-in Julia set; if it does not, we call it a **co-root**.

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Lemma (Schleicher)

- *Every co-root is the landing point of exactly one dynamic ray, and this ray has the same exact period as the component.*
- *Every periodic Fatou component of period greater than 1 corresponding to an attracting/parabolic orbit has exactly one root. If the period of the component is even; then it has exactly $d-2$ co-roots; if the period is odd; it has exactly d co-roots. Every Fatou component of period 1 has exactly $d+1$ co-roots and no root.*

Transferring roots/co-roots from the dynamical plane to the parameter plane: the odd period case

- The **orbit portraits** of the **dynamical root/co-roots** of the **centre** remain stable throughout the hyperbolic component and extend continuously to the **root/co-roots arcs** (except the cusp points).

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- At each parabolic point, the continuation of one of the root or co-roots become **parabolic** implying that every arc on the boundary of the component corresponds to a unique dynamical root/co-root.
- At least $d+1$ arcs: The combinatorial rigidity of the centre [Po] ensures that each dynamical root/co-root 'has its own arc'.
- At most $d+1$ arcs: The combinatorial rigidity of the parabolics [HS] ensures that exactly one arc 'corresponds to' a given dynamical root/co-root.

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- The orbit portraits of the dynamical roots throughout the bifurcated hyperbolic component is constant and is of $(k, 2k, 2k)$ type.
- However, on the sub-arcs along which the bifurcation occurs, two of these rays land together and another lands separately proving the discontinuous parameter dependence of the landing points of dynamical rays.

Number of hyperbolic components

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- Using the main theorem, one can count the number of hyperbolic components of period $k(\neq 2)$ which is twice an odd integer: $s'_{d,k} = s_{d,k} + 2s_{d,\frac{k}{2}}$.

Further topological questions

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- Do the accumulation sets of the decorations in the following figure ‘overlap’? An example of such an overlap would, in turn, prove that the corresponding parameter ray strictly accumulates.

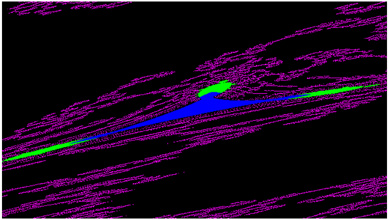

Figure: Zoom near a hyperbolic component of period 13 (blue) which shows the bifurcated components of period 26 (green) and the decorations coming out of them. These decorations accumulate on sub-arcs of the parabolic arcs.

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- Are the hyperbolic components of odd period (like the one in the previous figure) homeomorphic to the original multicorn via straightening?

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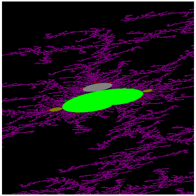
- Are the hyperbolic components of odd period (like the one in the previous figure) homeomorphic to the original multicorn via straightening?
- Are the hyperbolic components of even period (like the one in the following figure) homeomorphic to the original multibrot set via straightening?



Zoom near a hyperbolic component of even period, which resembles a baby mandelbrot set.

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