Distribution of postcritically finite polynomials

– Notes of my talk –

Thomas Gauthier (LAMFA – UPJV)

Tuesday June 11th, 2013

$\star$ *This is a joint work with Charles Favre* $\star$

Recall that a polynomial $P \in \mathbb{C}[z]$ of degree $d \geq 2$ is *postcritically finite* (*PCF*) if its postcritical set is finite, i.e. if

$$\bigcup_{n \geq 1} P^{\circ n}(C(P))$$

is a finite set.

This implies in particular that all $c \in C(P)$ is (pre)periodic under iteration by P.

Question. How do postcritically finite polynomials equidistribute in the space of all degree d polynomials when the (pre)period of each critical point tends to ∞?

We would like to give an answer to this question.

1 − The quadratic case

Let us set $P_c(z) := z^2 + c$ for $c \in \mathbb{C}$. The *filled Julia set* of P_c is

$$\mathcal{K}_c := \{z \in \mathbb{C} \mid (P_c^{\circ n}(z)) \text{ is bounded}\}.$$

The important picture appearing in the parameter space of the family $(P_c)_{c\in\mathbb{C}}$ is the *Mandelbrot set*

$$\begin{aligned} \mathcal{M}_2 &:= \{c \in \mathbb{C} \mid (P_c^{\circ n}(0)) \text{ is bounded}\} \\ &= \{c \in \mathbb{C} \mid \mathcal{K}_c \text{ is connected}\} . \end{aligned}$$

Let $\mathrm{Per}(n) := \{c \in \mathbb{C} \mid P_c^{\circ n}(0) = 0\}$.

Fact. *1. $c_0 \in \mathbb{C} \setminus \partial\mathcal{M}_2$ if and only if as a family of holomorphic maps of c, $\{c \mapsto P_c^{\circ n}(0)\}_{n\geq 1}$ is a normal family in a neigh. of c_0, i.e. $\partial\mathcal{M}_2$ is the* bifurcation locus,
2. $\partial\mathcal{M}_2 \subset \overline{\bigcup_{n\geq 1} \mathrm{Per}(n)}$.

Proof. Montel's Theorem. □

Question. How to quantify this approximation?

Recall that the subharmonic function

$$g_{\mathcal{M}_2}(c) := g_c(c) = \lim_n 2^{-n} \log^+ |P_c^{\circ n}(c)|$$

is the *Green function* of $\mathcal{M}_2$ and that

$$\mathcal{M}_2 = \{g_{\mathcal{M}_2}(c) = 0\} \ .$$

Let $\mu_{\text{bif}} := dd^c g_{\mathcal{M}_2}$, then $\text{supp}(\mu_{\text{bif}}) = \partial\mathcal{M}_2$. μ_{bif} is the *bifurcation, measure*.

Theorem (Levin)**.** *The sequence*

$$\mu_n := \frac{1}{2^{n-1}} \sum_{P_c^{\circ n}(0)=0} \delta_c$$

converges to μ_{bif} *in the weak topology of measures.*

Proof of Thm Levin. Since $P_c^n(0) = 0$ has simple roots (Douady-Hubbard),

$$\mu_n = dd^c \log |P_c^{\circ n}(0)| \ .$$

We then apply Hartogs' Lemma and get $\log |P_c^{\circ n}(0)| \to g_{\mathcal{M}2}(c)$ in L^1_{loc}. $\square$

The "*global method*" (Baker-H'sia) allows to get an estimate on the speed of convergence:

Theorem (Favre-Rivera-Letelier). *There exists $C > 0$, s.t. for $\varphi \in \mathcal{C}^1_c(\mathbb{C})$ and $n \geq 1$,*

$$\left| \int \varphi \, \mu_n - \int \varphi \, \mu_{\text{bif}} \right| \leq C \left(\frac{n}{2^n} \right)^{1/2} \|\varphi\|_{\mathcal{C}^1} \ .$$

2 – The cubic case

Let us set $P_{c,a}(z) := \frac{1}{3}z^3 - \frac{c}{2}z^2 + a^3$ for $(c,a) \in \mathbb{C}^2$. How to generalize $\mathcal{M}_2$?

Three different solutions:

1. The non-escaping set of $c_1 := 0$

$$\begin{aligned}\Gamma_1 &:= \{(c,a) \in \mathbb{C}^2 \mid (P_{c,a}^{\circ n}(0)) \text{ is bounded}\} \\ &= \{(c,a) \in \mathbb{C}^2 \mid g_{c,a}(c_1) = 0\}\ ,\end{aligned}$$

2. The non-escaping set of $c_2 := c$

$$\begin{aligned}\Gamma_2 &:= \{(c,a) \in \mathbb{C}^2 \mid (P_{c,a}^{\circ n}(c)) \text{ is bounded}\} \\ &= \{(c,a) \in \mathbb{C}^2 \mid g_{c,a}(c_2) = 0\}\ ,\end{aligned}$$

3. The connectedness locus

$$\begin{aligned}\mathcal{M}_3 &:= \{(c,a) \in \mathbb{C}^2 \mid \mathcal{K}_{c,a} \text{ is connected}\} \\ &= \{(c,a) \in \mathbb{C}^2 \mid \max_{i=1,2}\{g_{c,a}(c_i)\} = 0\} \\ &= \Gamma_1 \cap \Gamma_2\ .\end{aligned}$$

2.1 – Cases 1 and 2

The set $\partial\Gamma_1$ (resp. $\partial\Gamma_2$) is the bifurcation locus of the critical point c_1 (resp. c_2).

Theorem (Branner-Hubbard)**.** *$\mathcal{M}_3$ is a compact subset of $\mathbb{C}^2$.*

Let $\mathrm{Per}_1(n) := \{(c,a) \in \mathbb{C}^2 \,|\, P_{c,a}^{\circ n}(c_1) = c_1\}$ and $\mathrm{Per}_2(n) := \{(c,a) \in \mathbb{C}^2 \,|\, P_{c,a}^{\circ n}(c_2) = c_2\}$.

Recall that a closed positive $(1,1)$-current can be seen as a degenerate metric (as well as a positive measure is a degenerate volume form).

Let $T_i := dd^c g_{c,a}(c_i)$ for $i = 1, 2$ so that $\mathrm{supp}(T_i) = \partial\Gamma_i$. Generalizing (non-trivially) the approach of Levin one can prove the following.

Theorem (Dujardin-Favre)**.** *The sequence $3^{-n}[\mathrm{Per}_i(n)]$ converges to T_i for $i = 1, 2$.*

2.2 – Case 3

Let $\mathrm{Per}(n,m) := \mathrm{Per}_1(n) \cap \mathrm{Per}_2(m)$.

What plays here the role of $\partial\mathcal{M}_2$ is $\partial_{\mathrm{Sh}}\mathcal{M}_3$:

Theorem (Bassanelli-Berteloot, D-F)**.**

$$\partial_{\mathrm{Sh}}\mathcal{M}_3 \subset \overline{\bigcup_{n,m\geq 1} \mathrm{Per}(n,m)}.$$

We generalize μ_{bif} by setting (B-B, D-F)

$$\begin{aligned}\mu_{\mathrm{bif}} &:= T_1 \wedge T_2 \\ &= (dd^c g_{\mathcal{M}_3}(c,a))^2 \ ,\end{aligned}$$

where $g_{\mathcal{M}_3}(c,a) := \max_{i=1,2}\{g_{c,a}(c_i)\}$. It is known that (D-F)

$$\mathrm{supp}(\mu_{\mathrm{bif}}) = \partial_{\mathrm{Sh}}\mathcal{M}_3 \ .$$

Our main result is the following.

Theorem 1 (Favre-G.)**.** *Let $n_k \neq m_k$, with $n_k, m_k \to \infty$ as $k \to \infty$. Then the measures*

$$\mu_k := \frac{1}{3^{n_k+m_k}} \sum_{(c,a)\in \mathrm{Per}(n_k,m_k)} \delta_{c,a}$$

converge to μ_{bif}.

Global method

Our proof relies on Yuan's equiditribution of points of small heights.

Let $\mathcal{P} := \{p \geq 2 \text{ prime number}\}$. For $p \in \mathcal{P} \cup \{\infty\}$, we let $|\cdot|_p$ be

$$|\cdot|_p := \begin{cases} p-\text{adic norm} & \text{if } p \in \mathcal{P} \ , \\ \text{complex norm} & \text{if } p = \infty \ . \end{cases}$$

The *naive height* $h : \bar{\mathbb{Q}}^2 \to \mathbb{R}_+$ is

$$h(x) := \sum_{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} \sum_{\mathcal{P} \cup \{\infty\}} \frac{\log^+ \max_i \{|\sigma(x_i)|_p\}}{\deg(x)} \ .$$

Let $g_{c,a,p}(z) := \lim_n 3^{-n} \log^+ |P_{c,a}^{\circ n}(z)|_p$ and

$$g_{\mathcal{M}_3,p}(c,a) := \max\{g_{c,a,p}(0), g_{c,a,p}(c)\}.$$

We define a *critical height* $H : \bar{\mathbb{Q}}^2 \to \mathbb{R}_+$:

$$H(c,a) := \sum_{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} \sum_{\mathcal{P} \cup \{\infty\}} \frac{g_{\mathcal{M}_3,p}(\sigma(c,a))}{\deg(c,a)} \ .$$

Theorem (Yuan). *Let $(F_k) \subset \bar{\mathbb{Q}}^2$ be a sequence of finite sets such that:*
(1) F_k is Galois-invariant,
(2) $H|_{F_k} \equiv 0$ for all k,
(3) For any algebraic curve $Z \subset \mathbb{C}^2$,

$$\frac{\mathrm{Card}(F_k \cap Z)}{\mathrm{Card}(F_k)} \longrightarrow 0 \ .$$

Then in the sense of measures on $\mathbb{C}^2$:

$$\mu_k := \frac{1}{\mathrm{Card}(F_k)} \sum_{(c,a) \in F_k} \delta_{c,a} \longrightarrow \mu_{\mathrm{bif}} \ .$$

Let us set

$$F_k := \mathrm{Per}(n_k, m_k) \ .$$

Remark that $P_{c,a}^{\circ n}(c_i) - c_i \in \mathbb{Q}[c,a]$, i.e. that $\mathrm{Per}(n,m) \subset \bar{\mathbb{Q}}^2$ and F_k is Galois-invariant. Moreover, $H|_{F_k} = 0$ for all k. So (1) and (2) hold.

It remains to prove that (3) holds and that $\mathrm{Card}(F_k) \sim 3^{n_k + m_k}$ as $k \to \infty$.

Remark that there exists plenty of curve $Z \subset \mathbb{C}^2$ containing infinitely many $(c, a) \in \mathbb{C}^2$ s.t. $P_{c,a}$ is *PCF* (e.g. $\{c = 0\}$). So it is not sufficient to prove that $\text{Card}(F_k) \to \infty$. We rely on Epstein's transversality theory. The asumption $n_k \neq m_k$ is made for applying his theory. Conjecturally, it is a non-necessary condition.

Theorem (Epstein). *Let $(c, a) \in F_k$, then* $\text{Per}_1(n_k)$ *and* $\text{Per}_2(m_k)$ *are smooth and transverse at (c, a).*

Then, by Bezout,

$$\text{Card}(F_k) = \deg(F_k) \sim 3^{n_k + m_k}$$

and $\text{Card}(F_k \cap Z)$ is at most

$$C \cdot \max\{\deg(\text{Per}_1(n_k)), \deg(\text{Per}_2(m_k))\} \text{ ,}$$

i.e. $C \cdot 3^{\max(n_k, m_k)} = o(3^{n-k+m_k})$.

Further results and open questions.

The same method (plus combinatoric tools developped by Kiwi) allows to prove that Misiurewicz parameters with critical points of prescribed perperiods (and periods tending to ∞) also equidistribute towards μ_{bif}. Another consequence is the following.

Let $\mathrm{Per}_n(w) = \{(c,a) \in \mathbb{C}^2$ s.t. $P_{c,a}$ has a cycle of exact period n and multiplier $w\}$.

Theorem 2 (Favre-G.). *Let $n_k \neq m_k$, with $n_k, m_k \to \infty$ as $k \to \infty$. Let also $(w_1, w_2) \in \mathbb{D}^2$. Then the measures*

$$\mu_k := \frac{1}{3^{n_k+m_k}} \sum_{\mathrm{Per}_{n_k}(w_1)\cap\mathrm{Per}_{m_k}(w_2)} \delta_{c,a}$$

converge to μ_{bif}.

Question. What happens in the case when $|w_1| \geq 1$ and/or $|w_2| \geq 1$?