# EXPLOSION POINTS FOR EXPONENTIAL MAPS

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# Outline

The study of the dynamics of holomorphic complex maps $f : \mathbb{C} \to \mathbb{C}$ was started in the early twentieth century by Pierre Fatou and Gaston Julia [1].

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They focused on studying the behavior of the points on $\mathbb{C}$ under iteration of the function $f$, and they divided the plane into two disjoint invariant sets[1].

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Most of the interesting and mysterious dynamics appears on the Julia set $J(f)$ . Due to that most of the study of the dynamics of holomorphic functions centers on the structure of the Julia set.

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For certain parameters, including $a \in (-\infty, -1)$, the Julia set $J(f_a)$ of this family is well understood. It is an uncountable union of curves, each consisting of a finite endpoint and a ray that connects this endpoint to infinity.

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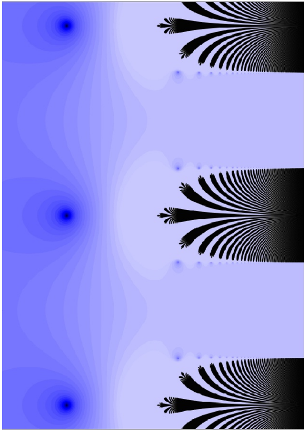

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### Theorem

*Let $a \in \mathbb{C}$. Then there is an invariant set $A \subset \tilde{E}(f_a)$ such that $\infty$ is an explosion point of $A \cup \{\infty\}$.*
*In particular, $\tilde{E}(f_a) \cup \{\infty\}$ is connected.*

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This model was developed by **Lasse Rempe-Gillen** [6] and will be defined later.
We proved first that the set of escaping endpoints of $\overline{X}$ satisfies Mayer's property (that it is totally disconnected, but connected when joined infinity). This uses the concept of a *Lelek fan*. We then transfer the result to the complex plane by using a general conjugacy result from [6]

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For a continuum $S$ if for every closed, connected subsets $A$ and $B$ of $S$ such that $S = A \cup B$ and $A \cap B$ is connected then $S$ is called unicoherent continuum.
If every closed, connected subset of $S$ is a unicoherent then $S$ is called a hereditarily unicoherent continuum.
In other words, a continuum $S$ is hereditarily unicoherent if and only if it does not contain any subset that disconnected the sphere $\widehat{\mathbb{C}}$.

**Definition**

*A dendroid is an arcwise connected hereditarily unicoherent continuum.*

A dendroid is a uniquely arcwise connected, otherwise it could be not hereditarily unicoherent.

A point $z_0$ is said to be a ramification point of the dendroid $S$ if there are three arcs $az_0, bz_0$ and $cz_0$ in $S$ and the intersection of any two of these three arcs is only the point $\{z_0\}$.

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An endpoint of a fan is a point that is the endpoint of each arc containing it.

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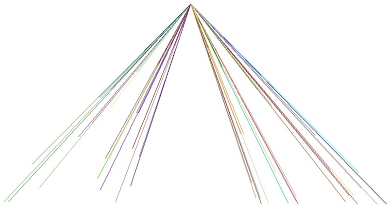

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### Theorem (Charatonik)

*If $Y$ is a smooth fan, different from an arc, then the following are equivalent:*

1. *the set of endpoints of $Y$ is dense;*
2. *the set of the endpoints of $Y$ together with the top is connected.*

The top point on the Lelek fan is called the explosion point of the set of endpoints of the fan. A point $m_0$ is called an *explosion point* of a topological space $M$ if $M$ is connected but $M \setminus \{m_0\}$ is totally separated.

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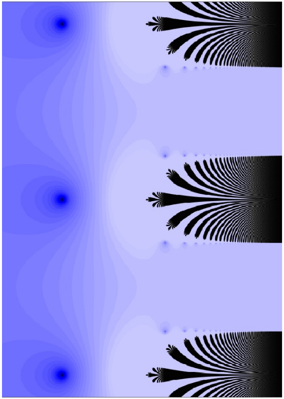
In our proof we are going to use Charatonik's theorem by showing that our chosen subset of the set of escaping endpoints satisfies 1 and hence it is homeomorphic to the Lelek fan and satisfies 2.

Back to our Julia set of the map $f_{-2}(z)$.



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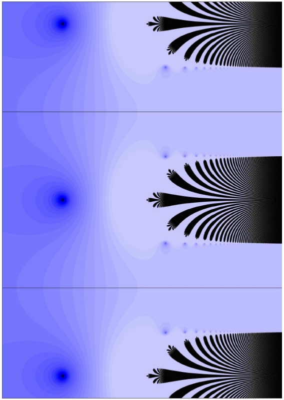

For each $z \in J(f_a)$ a unique sequence $\underline{s} = s_0 s_1 s_2 s_3 \ldots$,
$f_a^k(z) \in S_{s_k} \forall k$.

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and a function

$$\mathcal{F} : \mathbb{Z}^{\mathbb{N}_0} \times [0, \infty) \to \mathbb{Z}^{\mathbb{N}_0} \times \mathbb{R}; \mathcal{F}(\underline{s}, t) := (\sigma(\underline{s}), F(t) - 2\pi|s_1|).$$

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Then we set

$$\overline{X} := \{(\underline{s}, t) \in \mathbb{Z}^{\mathbb{N}_0} \times [0, \infty) : \mathcal{F}^n(\underline{s}, t) \in \mathbb{Z}^{\mathbb{N}_0} \times [0, \infty) \text{ for all } n \geq 0\}.$$

For every sequence of integers $\underline{s}$, there is a real number $t_{\underline{s}}$ with $0 \leq t_{\underline{s}} \leq \infty$ such that

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The point $(\underline{s}, t_{\underline{s}})$ is called an endpoint. Moreover we define $E$ to be the set of all endpoints in $\overline{X}$.

A point $(\underline{s}, t) \in \overline{X}$ is called an escaping point if its second component is escaping to infinity under iteration of $\mathcal{F}$. As in [6] we will write $T$ as the projection to the second component; i.e. $T(\underline{s}, t) = t$.

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We define $X$ to be the set of all escaping points of $\overline{X}$,

$$X := \{(\underline{s}, t) \in \overline{X} : T(\mathcal{F}^n(\underline{s}, t)) \to \infty\}.$$

Therefore the endpoints which escape are called the escaping endpoints of $\overline{X}$ and denoted

$$\tilde{E} := E \cap X.$$

Now I will define some subsets of $\tilde{E}$ and prove some lemmas about these subsets to prove our theorem.

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Now I will define some subsets of $\tilde{E}$ and prove some lemmas about these subsets to prove our theorem.

A sequence $\underline{s}_0 \in \mathbb{Z}^{\mathbb{N}_0}$ is called an exponentially bounded address if $t_{\underline{s}} < \infty$.

Furthermore, a sequence $\underline{s}$ is called fast if $(\underline{s}, t_{\underline{s}}) \in X$.

Let $\underline{s}^0$ be an exponentially bounded address and define $\mathcal{A}(\underline{s}^0)$ and $\overline{X}(\underline{s}^0)$ as :

$$\mathcal{A}(\underline{s}^0) := \{\underline{s} \in \mathbb{Z}^{\mathbb{N}_0} : |s_j| \geq |s_j^0| \text{ for all } j\},$$

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**Lemma**

*Let $\underline{s}_0$ be fast address. Then the space $\overline{X}(\underline{s}^0)$ is a smooth fan.*

For an exponentially bounded address $\underline{s}$ define $t^*_{\underline{s}}$ as

$$t^*_{\underline{s}} := \sup_{k\geq 1} F^{-k}(2\pi|s_k|)$$

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### Lemma

*For any $x \in \overline{X}(\underline{s}^0)$ where $x := (\underline{s}, t)$ let $x^n := (\underline{s}^n, t_{\underline{s}_n}) \in \overline{X}(\underline{s}^0)$ , $c > 0$ and a sequence $k_n \to \infty$ such that:*

1. $s^n_j = s_j$ *for all* $j \leq k_n$ *and* $|s^n_i| \geq |s_i|$ *for all* $i > k_n$.
2. $|t^*_{\sigma^{k_n}(\underline{s}^n)} - T(F^{k_n}(x))| \leq c$.

*then* $(\underline{s}^n, t_{\underline{s}^n}) \to x$.

**Theorem**

*The set of endpoints $E(\overline{X}(\underline{s}^0))$ of the space $\overline{X}(\underline{s}^0)$ is dense.*

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### Proposition

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Hence we could easily prove the following theorem:

**Theorem**

*Let $\tilde{E}$ be the set of escaping endpoints in $\overline{X}$:*

$$\tilde{E} := \{(\underline{s}, t_{\underline{s}}) : \underline{s} \text{ is fast}\}.$$

*Then*

1. *$\tilde{E}$ is totally disconnected.*
2. *$\tilde{E} \cup \{\infty\}$ is connected.*

The result:

**Theorem**

*Let $a \in \mathbb{C}$. Then there is an invariant set $A \subset \tilde{E}(f_a)$ such that $\infty$ is an explosion point of $A \cup \{\infty\}$. In particular, $\tilde{E}(f_a) \cup \{\infty\}$ is connected.*

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*Thanks for listening.*