On the construction of entire functions in the Speiser class

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In other words: given a tract G, find an entire function in class $\mathcal{S}$ which has only this tract.
One possibility to construct functions in class $\mathcal{S}$ with a given property is *quasiconformal folding*, a method introduced by C. Bishop in 2011.

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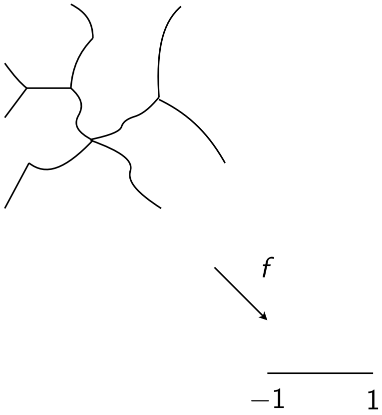

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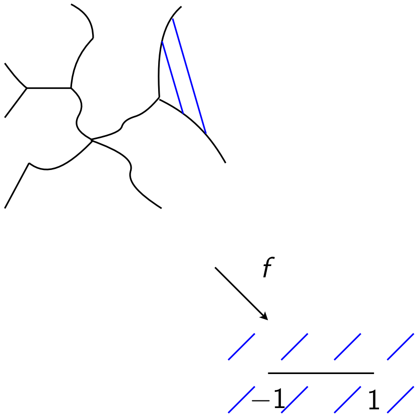

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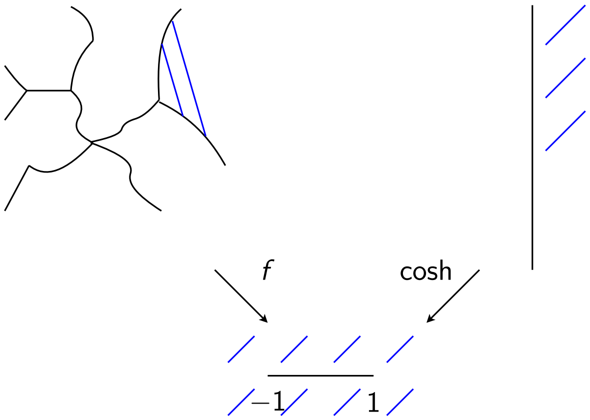

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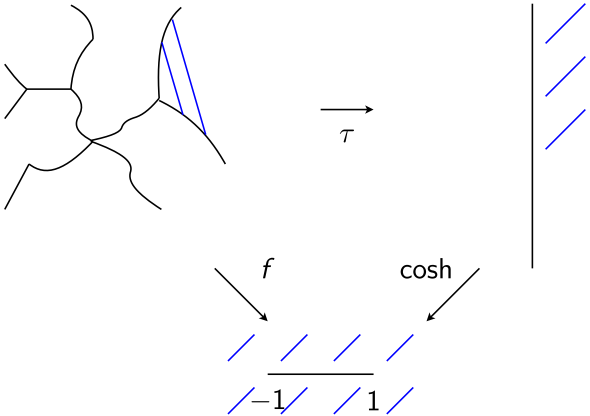

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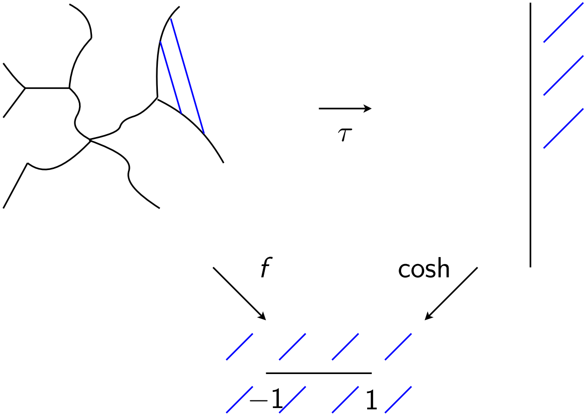

Reverse this procedure!

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- Apply the measurable Riemann mapping theorem.

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- for non-adjacent edges e and f, $\frac{\mathrm{diam}(e)}{\mathrm{dist}(e,f)}$ is uniformly bounded.

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- R components: unbounded simply connected components (not necessarily Jordan domains), which are mapped onto the right half-plane (our illustration only used such components).
- L components: unbounded Jordan domains, which are mapped onto the left half-plane (these components will assign asymptotic values to f).
- D components: bounded Jordan domains, which are mapped onto $\mathbb{D}$ (these components will assign critical points of arbitrary high order to f).

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Then there is an entire function f and a quasiconformal map ϕ of the plane so that $f \circ \phi = \sigma \circ \tau$ off $T(r_0)$ (a neighbourhood of T). The only singular values of f are ± 1 (critical values coming from the vertices of T) and the critical values and singular values assigned by the D and L components.

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  - Merenkov's results on functions in class $\mathcal{S}$ of arbitrary order of growth.
  - Every bounded, countable subset of $\mathbb{C}$ (which contains at least two points) can be the singular set of an entire function in class $\mathcal{B}$.

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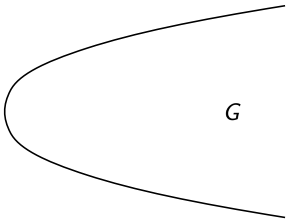
Remark

The regularity conditions are satisfied e.g. for x^ε if $\frac{1}{2} \leq \varepsilon < 1$.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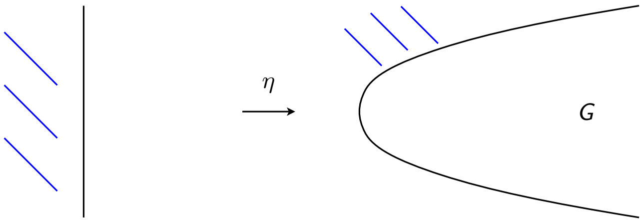
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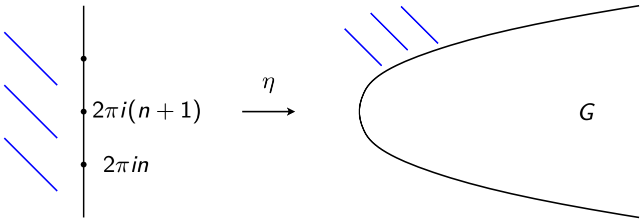
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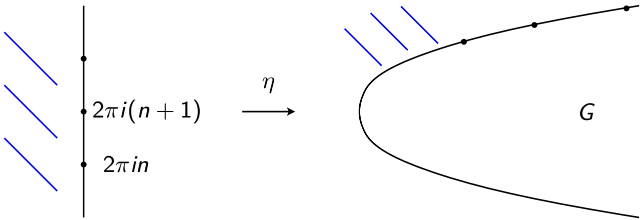
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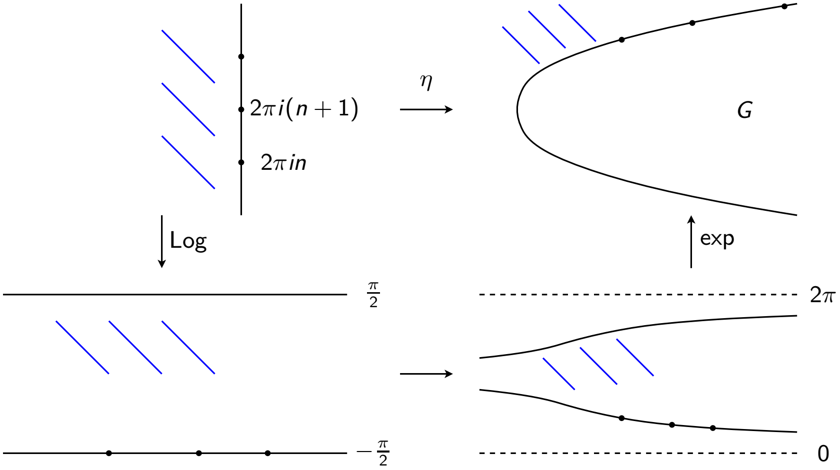
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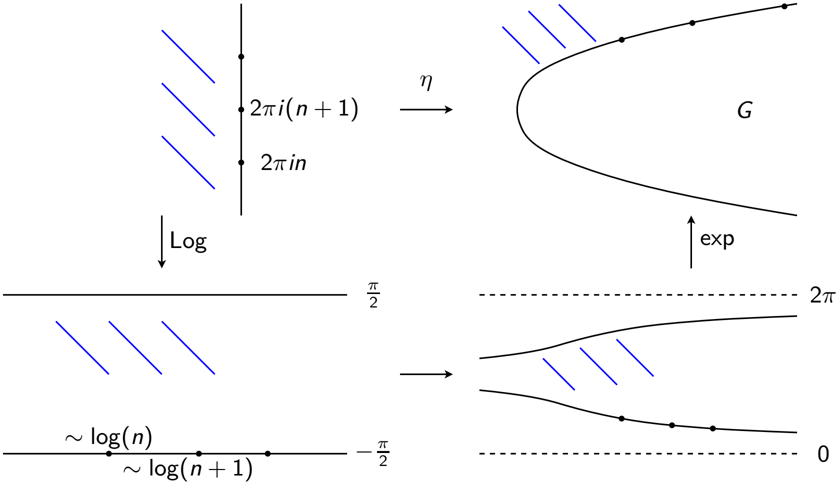
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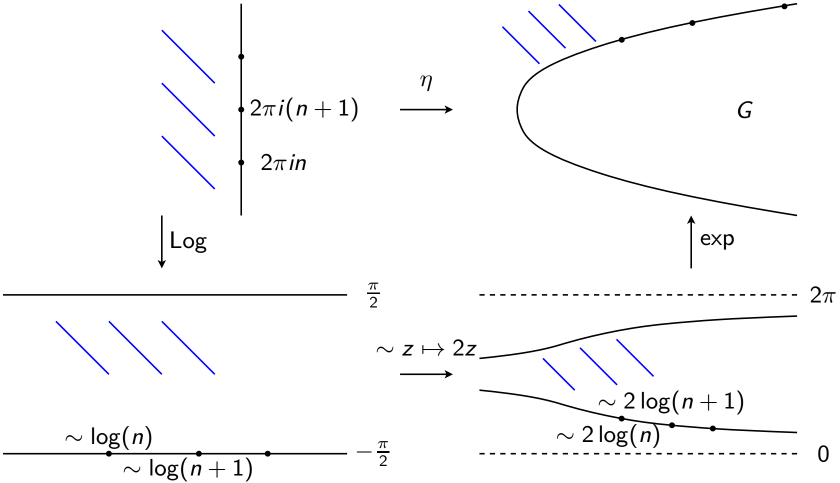

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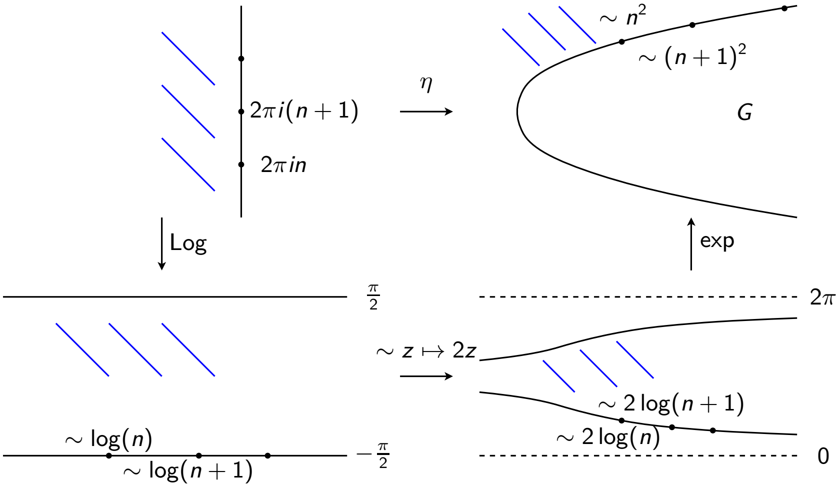
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Let $\Omega = \{x + iy \ : \ |y| < \psi(x)\}$ *be a horizontal strip with some non-negative function* $\psi : \mathbb{R} \to \mathbb{R}$. *Let* σ *map* Ω *conformally onto* $\{x + iy \ : \ |y| < \frac{\pi}{2}\}$ *such that* $\mathrm{Re}(\sigma(z)) \to \pm\infty$ *as* $\mathrm{Re}(z) \to \pm\infty$. *Let* $\beta(x) = \inf_{|y|<\psi(x)} \mathrm{Re}(\sigma(x + iy))$ *and* $\alpha(x) = \sup_{|y|<\psi(x)} \mathrm{Re}(\sigma(x + iy))$. *If* $\int_{x_1}^{x_2} \frac{\mathrm{d}t}{\psi(t)} > 4$

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$$\beta(x_2) - \alpha(x_1) \geq \frac{\pi}{2} \int_{x_1}^{x_2} \frac{\mathrm{d}t}{\psi(t)} - 4\pi.$$

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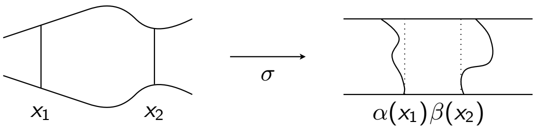

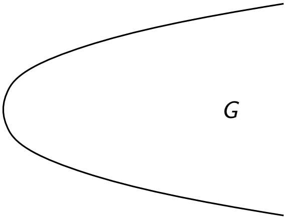
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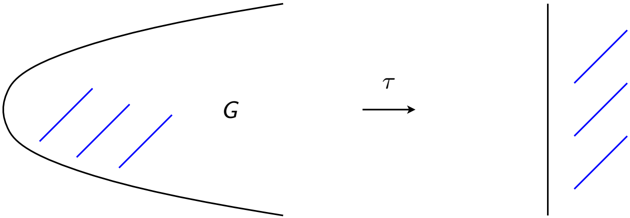
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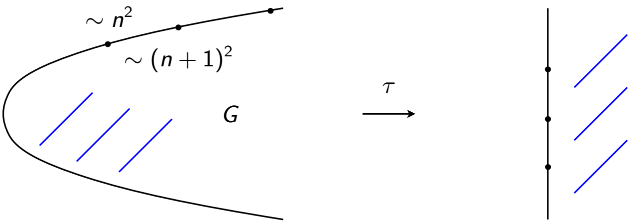
$\sim n^2$
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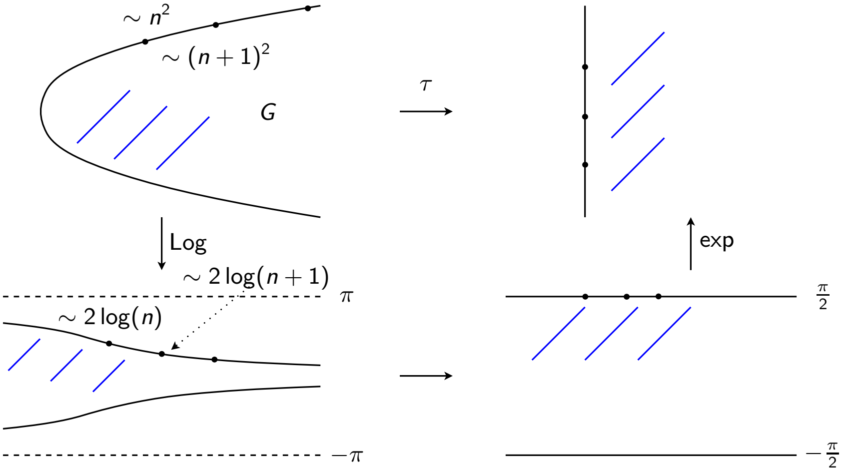

$\sim n^2$
$\sim (n+1)^2$
G
τ
Log
exp
$\sim 2\log(n+1)$
π
$\sim 2\log(n)$
$-\pi$
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Remark

In many cases we even get $f \sim \exp \circ \omega$.

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  - construct f in class S with $\dim_H(I(f)) = 1$.

Thank you very much for your attention.