A combinatorial point of view of some dynamic problems.

J. Tomasini

10/06/2013

Sommaire

Introduction

We consider the differential equation:

$$\dot{z} = P(z), \quad P \in \mathbb{C}[X].$$

We denote by ξ_P the poynomial vector field defined by P.

Introduction

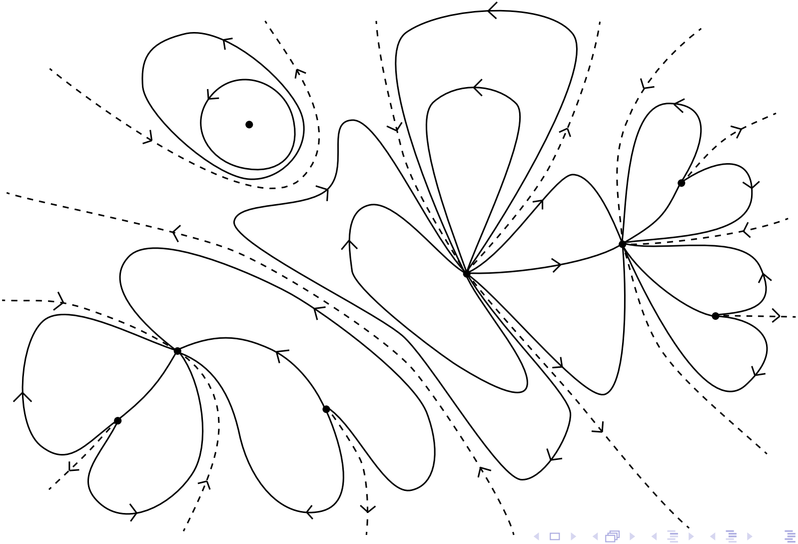
We consider the differential equation:

$$\dot{z} = P(z), \quad P \in \mathbb{C}[X].$$

We denote by ξ_P the poynomial vector field defined by P.
Let ζ be a root of the polynomial P. Then the vector field ξ_P associated to P admits an equilibrium point (or singularity) at the point ζ, and this singularity can be of four different types:

- ζ is a source if $Re(P'(\zeta)) > 0$.
- ζ is a sink if $Re(P'(\zeta)) < 0$.
- ζ is a center if $Re(P'(\zeta)) = 0$ and $Im(P'(\zeta)) \neq 0$.
- ζ is a multiple equilibrium point of multiplicity $m \geq 2$ if $P'(\zeta) = \ldots = P^{(m-1)}(\zeta) = 0$ and $P^{(m)}(\zeta) \neq 0$.

Introduction



Bassins

A given singularity ζ determines the behavior of the solutions passing through a neighborhood of it. This zone of influence is called **bassin**, denoted by $\mathcal{B}(\zeta)$, and is defined as follows:

Bassins

A given singularity ζ determines the behavior of the solutions passing through a neighborhood of it. This zone of influence is called **bassin**, denoted by $\mathcal{B}(\zeta)$, and is defined as follows:

- If ζ is a source, $\mathcal{B}(\zeta) = \{z \in \mathbb{C} \mid \gamma(t, z) \to \zeta \text{ for } t \to -\infty\}$.

Bassins

A given singularity ζ determines the behavior of the solutions passing through a neighborhood of it. This zone of influence is called **bassin**, denoted by $\mathcal{B}(\zeta)$, and is defined as follows:

- If ζ is a source, $\mathcal{B}(\zeta) = \{z \in \mathbb{C} \mid \gamma(t, z) \to \zeta \text{ for } t \to -\infty\}$.
- If ζ is a sink, $\mathcal{B}(\zeta) = \{z \in \mathbb{C} \mid \gamma(t, z) \to \zeta \text{ for } t \to +\infty\}$.

Bassins

A given singularity ζ determines the behavior of the solutions passing through a neighborhood of it. This zone of influence is called **bassin**, denoted by $\mathcal{B}(\zeta)$, and is defined as follows:

- If ζ is a source, $\mathcal{B}(\zeta) = \{z \in \mathbb{C} \mid \gamma(t, z) \to \zeta \text{ for } t \to -\infty\}$.
- If ζ is a sink, $\mathcal{B}(\zeta) = \{z \in \mathbb{C} \mid \gamma(t, z) \to \zeta \text{ for } t \to +\infty\}$.
- If ζ is a center, $\mathcal{B}(\zeta) = \{\zeta\} \cup \{z \in \mathbb{C} \mid \gamma(., z) \text{ is periodic and } \zeta \text{ is in the bounded component of } \mathbb{C} \setminus \gamma(\mathbb{R}, z)\}$.

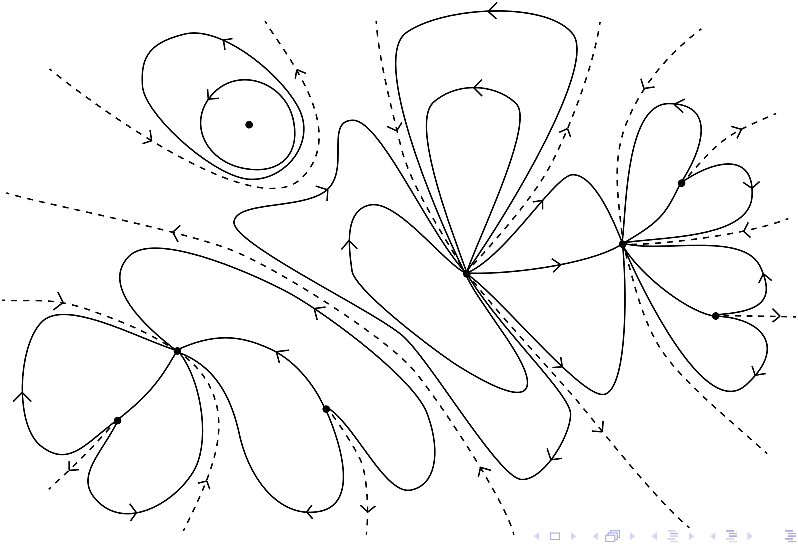
Bassins

A given singularity ζ determines the behavior of the solutions passing through a neighborhood of it. This zone of influence is called **bassin**, denoted by $\mathcal{B}(\zeta)$, and is defined as follows:

- If ζ is a source, $\mathcal{B}(\zeta) = \{z \in \mathbb{C} \mid \gamma(t, z) \to \zeta \text{ for } t \to -\infty\}$.
- If ζ is a sink, $\mathcal{B}(\zeta) = \{z \in \mathbb{C} \mid \gamma(t, z) \to \zeta \text{ for } t \to +\infty\}$.
- If ζ is a center, $\mathcal{B}(\zeta) = \{\zeta\} \cup \{z \in \mathbb{C} \mid$ $\gamma(., z)$ is periodic and ζ is in the bounded component of $\mathbb{C} \setminus \gamma(\mathbb{R}, z)\}$.
- If ζ is a multiple equilibrium point, $\mathcal{B}(\zeta) = \mathcal{B}_\alpha(\zeta) \cup \mathcal{B}_\omega(\zeta) \cup \{\zeta\}$, where

$$\begin{aligned}
\mathcal{B}_\alpha(\zeta) &= \{z \neq \zeta \mid \gamma(t, z) \to \zeta \text{ for } t \to -\infty\}. \\
\mathcal{B}_\omega(\zeta) &= \{z \neq \zeta \mid \gamma(t, z) \to \zeta \text{ for } t \to +\infty\}.
\end{aligned}$$

Bassins



Separatrix

there exist $2n-2$ solutions γ_l, with $l \in \{0, \ldots, 2n-3\}$, of the polynomial differential equation $\dot{z} = P(z)$ defined in a neighborhood of infinity and asymptotic to the ray $t.\delta_l$ for t large enough, where δ_l is the consecutive $2(n-1)$-th roots of unity.
we call separatrices of the vector field ξ_P, noted s_l, the maximal trajectories of ξ_P which coincide with the particular solutions γ_l.
We distinguish three types of separatrices:

Separatrix

there exist $2n-2$ solutions γ_l, with $l \in \{0, \ldots, 2n-3\}$, of the polynomial differential equation $\dot{z} = P(z)$ defined in a neighborhood of infinity and asymptotic to the ray $t.\delta_l$ for t large enough, where δ_l is the consecutive $2(n-1)$-th roots of unity.
we call separatrices of the vector field ξ_P, noted s_l, the maximal trajectories of ξ_P which coincide with the particular solutions γ_l.
We distinguish three types of separatrices:

- an outgoing separatrix.

Separatrix

there exist $2n-2$ solutions γ_l, with $l \in \{0, \ldots, 2n-3\}$, of the polynomial differential equation $\dot{z} = P(z)$ defined in a neighborhood of infinity and asymptotic to the ray $t.\delta_l$ for t large enough, where δ_l is the consecutive $2(n-1)$-th roots of unity. we call separatrices of the vector field ξ_P, noted s_l, the maximal trajectories of ξ_P which coincide with the particular solutions γ_l. We distinguish three types of separatrices:

- an outgoing separatrix.
- an incoming separatrix.

Separatrix

there exist $2n-2$ solutions γ_l, with $l \in \{0, \ldots, 2n-3\}$, of the polynomial differential equation $\dot{z} = P(z)$ defined in a neighborhood of infinity and asymptotic to the ray $t.\delta_l$ for t large enough, where δ_l is the consecutive $2(n-1)$-th roots of unity. we call separatrices of the vector field ξ_P, noted s_l, the maximal trajectories of ξ_P which coincide with the particular solutions γ_l. We distinguish three types of separatrices:

- an outgoing separatrix.
- an incoming separatrix.
- a homoclinic separatrix.

Separatrix

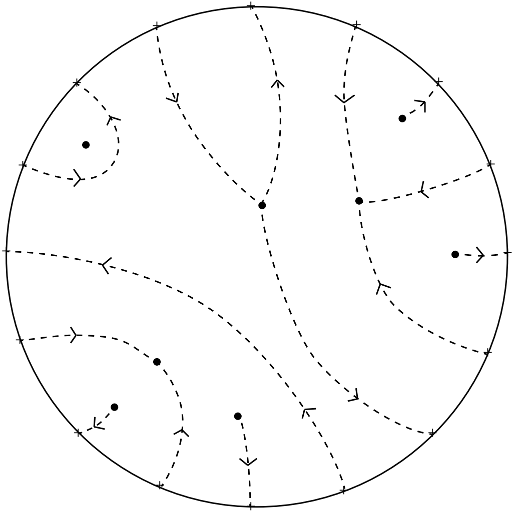
there exist $2n-2$ solutions γ_l, with $l \in \{0,\ldots,2n-3\}$, of the polynomial differential equation $\dot{z} = P(z)$ defined in a neighborhood of infinity and asymptotic to the ray $t.\delta_l$ for t large enough, where δ_l is the consecutive $2(n-1)$-th roots of unity. we call separatrices of the vector field ξ_P, noted s_l, the maximal trajectories of ξ_P which coincide with the particular solutions γ_l. We distinguish three types of separatrices:

- an outgoing separatrix.
- an incoming separatrix.
- a homoclinic separatrix.

First Modelisation

The **separatrix graph** Γ_P allows to identify the topological structure of polynomial vector fields.

Separatrix



Equivalence relation

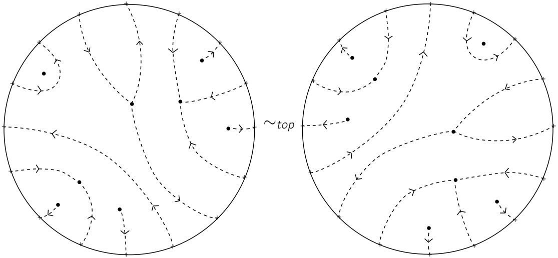
Definition

Let P, Q be two monic, centered polynomials of degree n, and Γ_P, Γ_Q be their respective separatrix graphs. We say that P is topologically equivalent to Q, denoted by $P \sim_{top} Q$, if there exists an isotopy $h : \overline{\mathbb{D}} \times [0,1] \to \overline{\mathbb{D}}$ that sends separatrices of Γ_P to separatrices of Γ_Q.

Equivalence relation

Definition

Let P, Q be two monic, centered polynomials of degree n, and Γ_P, Γ_Q be their respective separatrix graphs. We say that P is topologically equivalent to Q, denoted by $P \sim_{top} Q$, if there exists an isotopy $h : \overline{\mathbb{D}} \times [0,1] \to \overline{\mathbb{D}}$ that sends separatrices of Γ_P to separatrices of Γ_Q.



Zones

Let Z be a connected component of $\overline{\mathbb{D}} \setminus \Gamma_P$ (where the separatrix graph Γ_P is embedded in $\overline{\mathbb{D}}$). Such a component is called **zone** and can be of three different types:

Zones

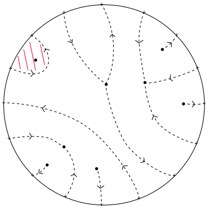
Let Z be a connected component of $\overline{\mathbb{D}} \setminus \Gamma_P$ (where the separatrix graph Γ_P is embedded in $\overline{\mathbb{D}}$). Such a component is called **zone** and can be of three different types:

1. a center zone.

Zones

Let Z be a connected component of $\overline{\mathbb{D}} \setminus \Gamma_P$ (where the separatrix graph Γ_P is embedded in $\overline{\mathbb{D}}$). Such a component is called **zone** and can be of three different types:

1. a center zone.



Zones

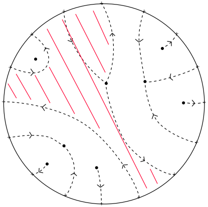
Let Z be a connected component of $\overline{\mathbb{D}} \setminus \Gamma_P$ (where the separatrix graph Γ_P is embedded in $\overline{\mathbb{D}}$). Such a component is called **zone** and can be of three different types:

1. a center zone.
2. a sepal zone.

Zones

Let Z be a connected component of $\overline{\mathbb{D}} \setminus \Gamma_P$ (where the separatrix graph Γ_P is embedded in $\overline{\mathbb{D}}$). Such a component is called **zone** and can be of three different types:

1. a center zone.
2. a sepal zone.



Zones

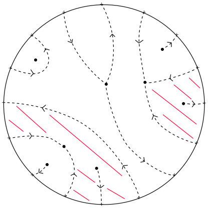
Let Z be a connected component of $\overline{\mathbb{D}} \setminus \Gamma_P$ (where the separatrix graph Γ_P is embedded in $\overline{\mathbb{D}}$). Such a component is called **zone** and can be of three different types:

1. a center zone.
2. a sepal zone.
3. an $\alpha\omega$-zone.

Zones

Let Z be a connected component of $\overline{\mathbb{D}} \setminus \Gamma_P$ (where the separatrix graph Γ_P is embedded in $\overline{\mathbb{D}}$). Such a component is called **zone** and can be of three different types:

1. a center zone.
2. a sepal zone.
3. an $\alpha\omega$-zone.



Zones

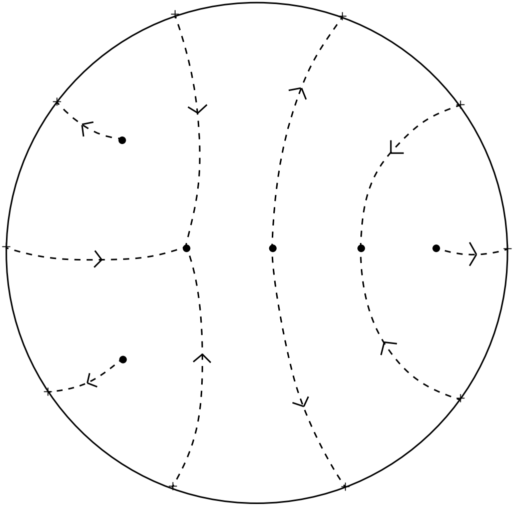
Let Z be a connected component of $\overline{\mathbb{D}} \setminus \Gamma_P$ (where the separatrix graph Γ_P is embedded in $\overline{\mathbb{D}}$). Such a component is called **zone** and can be of three different types:

1. a center zone.
2. a sepal zone.
3. an $\alpha\omega$-zone.

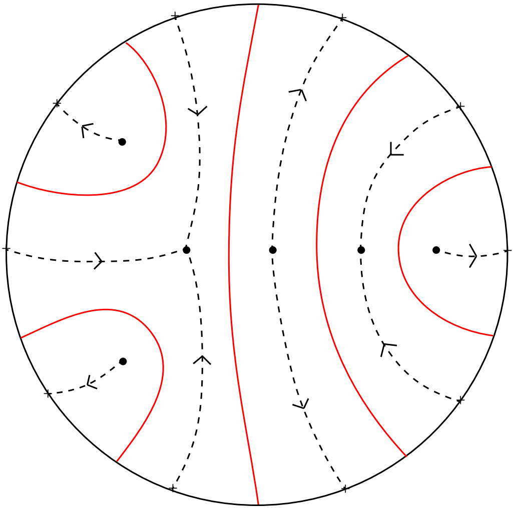
Second Modelisation

The **transversal graph** Σ_P.

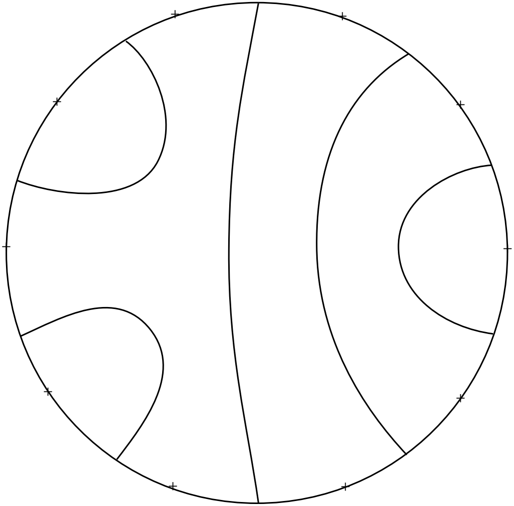
Zones



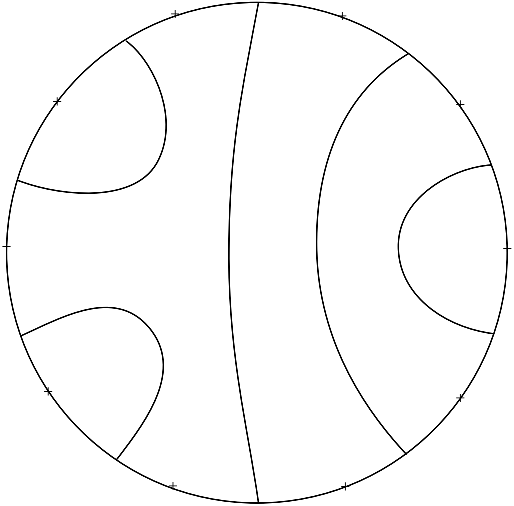
Zones



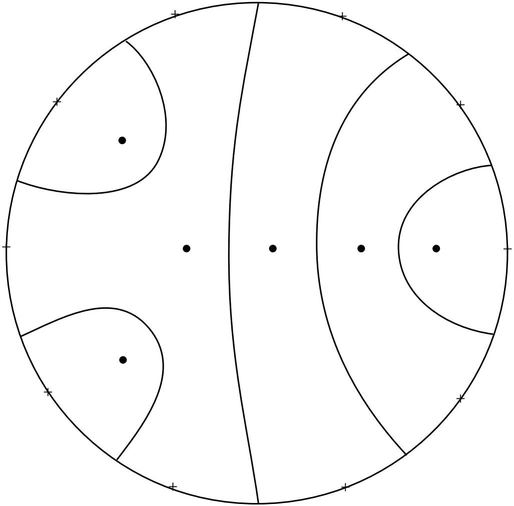
Zones



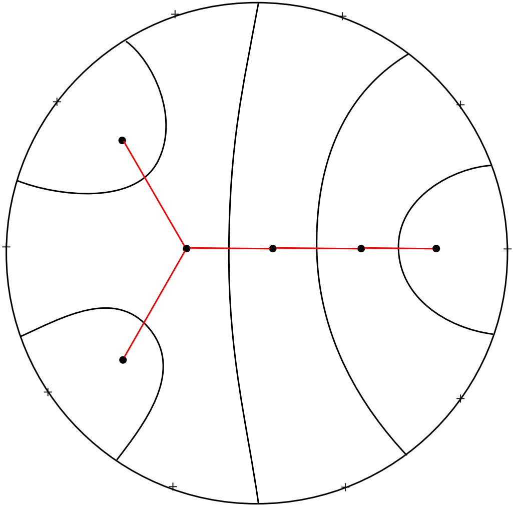
Tree



Tree



Tree



Enumeration

$$\sigma_n = \frac{1}{2n}\left[\frac{1}{n+1}\binom{2n}{n} + \sum_{\substack{l\geq 2\\ l|n}} \varphi(l)\binom{2n/l}{n/l} + \begin{cases} \binom{n}{\frac{n-1}{2}} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}\right].$$

Sommaire

treelike equivalence relation

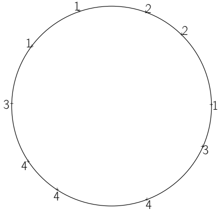
Definition

A **treelike equivalence relation** on the circle is a closed equivalence relation such that for any two distinct equivalence classes, their convex hulls in the unit disk are disjoint.

treelike equivalence relation

Definition

A **treelike equivalence relation** on the circle is a closed equivalence relation such that for any two distinct equivalence classes, their convex hulls in the unit disk are disjoint.



Lamination

Let R be a treelike equivalence relation, there is an associated lamination $Lam(R)$ of the open disk, where the leaves of $Lam(R)$ consist of boundaries of convex hulls of equivalence classes intersected with the open disk.

Lamination

Let R be a treelike equivalence relation, there is an associated lamination $Lam(R)$ of the open disk, where the leaves of $Lam(R)$ consist of boundaries of convex hulls of equivalence classes intersected with the open disk.
The regions bounded by leaves are called **gaps**. There are two types of gaps:

Lamination

Let R be a treelike equivalence relation, there is an associated lamination $Lam(R)$ of the open disk, where the leaves of $Lam(R)$ consist of boundaries of convex hulls of equivalence classes intersected with the open disk.
The regions bounded by leaves are called **gaps**. There are two types of gaps:

- collapsed gaps.

Lamination

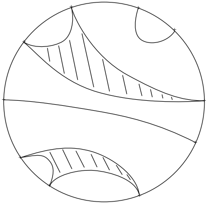
Let R be a treelike equivalence relation, there is an associated lamination $Lam(R)$ of the open disk, where the leaves of $Lam(R)$ consist of boundaries of convex hulls of equivalence classes intersected with the open disk.
The regions bounded by leaves are called **gaps**. There are two types of gaps:

- collapsed gaps.
- intact gaps.

Lamination

Let R be a treelike equivalence relation, there is an associated lamination $Lam(R)$ of the open disk, where the leaves of $Lam(R)$ consist of boundaries of convex hulls of equivalence classes intersected with the open disk.
The regions bounded by leaves are called **gaps**. There are two types of gaps:

- collapsed gaps.
- intact gaps.



Primitive majors

A lamination is **degree d invariant** if

Primitive majors

A lamination is **degree d invariant** if

i. If there is a leaf with endpoints x and y, then either $x^d = y^d$ or there is a leaf with endpoints x^d and y^d.

Primitive majors

A lamination is **degree d invariant** if

i. If there is a leaf with endpoints x and y, then either $x^d = y^d$ or there is a leaf with endpoints x^d and y^d.

ii. If there is a leaf with endpoints x and y, there is a set of d disjoint leaves with one endpoints in $x^{1/d}$ and the other endpoint in $y^{1/d}$.

Primitive majors

A lamination is **degree d invariant** if

i. If there is a leaf with endpoints x and y, then either $x^d = y^d$ or there is a leaf with endpoints x^d and y^d.
ii. If there is a leaf with endpoints x and y, there is a set of d disjoint leaves with one endpoints in $x^{1/d}$ and the other endpoint in $y^{1/d}$.

A **critical gap** is a gap that maps with degree greater than 1. The criticality of a gap is its degree minus 1.

Primitive majors

A lamination is **degree d invariant** if

i. If there is a leaf with endpoints x and y, then either $x^d = y^d$ or there is a leaf with endpoints x^d and y^d.
ii. If there is a leaf with endpoints x and y, there is a set of d disjoint leaves with one endpoints in $x^{1/d}$ and the other endpoint in $y^{1/d}$.

A **critical gap** is a gap that maps with degree greater than 1. The criticality of a gap is its degree minus 1.

Proposition

For any degree d invariant lamination λ, the total criticality of λ equals $d - 1$.

Primitive majors

Definition

A **major** for a degree d invariant lamination is the set of critical gaps.

Primitive majors

Definition

A **major** for a degree d invariant lamination is the set of critical gaps.

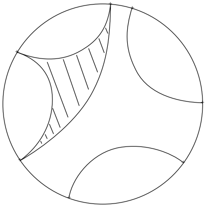
A major is called **primitive** if each (critical) gap is a polygon whose vertices are all identified by $z \mapsto z^d$. We denote by $PM(d)$ the set of all primitive degree d majors.

Primitive majors

Definition

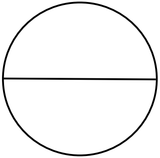
A **major** for a degree d invariant lamination is the set of critical gaps.

A major is called **primitive** if each (critical) gap is a polygon whose vertices are all identified by $z \mapsto z^d$. We denote by $PM(d)$ the set of all primitive degree d majors.

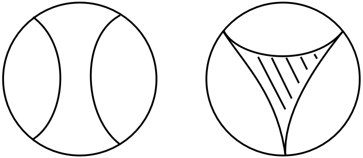


Primitive majors

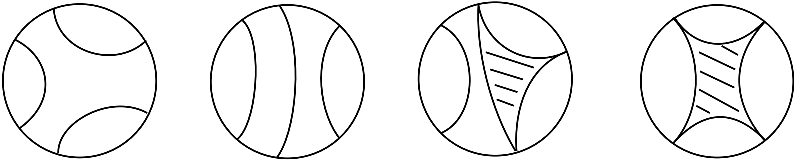
$d = 2$



$d = 3$

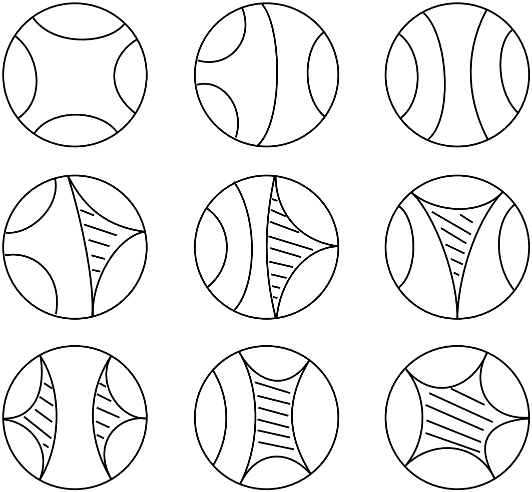


$d = 4$

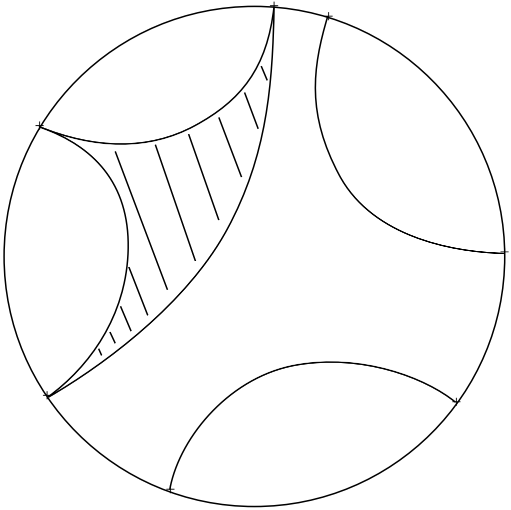


Primitive majors

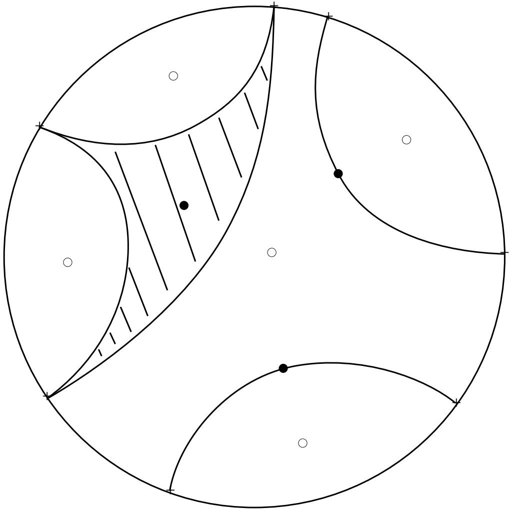
$d = 5$



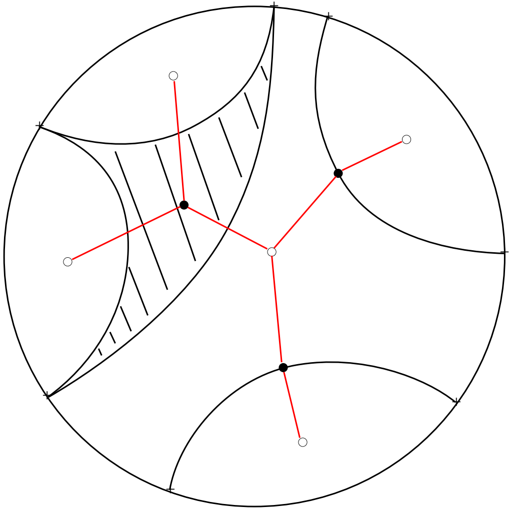
Bipartite tree



Bipartite tree



Bipartite tree



Enumeration

$$p_n^k = \frac{1}{n}\sum_{i=k}^{n-1} \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} n-1+i \\ i \end{pmatrix} \begin{pmatrix} n \\ n-1-i \end{pmatrix} (-1)^{n-1-i}$$

$$\tilde{p}_n^k = \frac{1}{n+k-1}\left[p_n^k + \sum_{\substack{l\geq 2 \\ l|n-1 \\ l|k}} \varphi(l)\left(\frac{n-1}{l}+1\right) p_{(n-1)/l+1}^{k/l} + \sum_{\substack{l\geq 2 \\ l|n \\ l|k-1}} \varphi(l)\left(\left(\frac{k-1}{l}+1\right) p_{n/l}^{(k-1)/l+1} + \left(\frac{n+k-1}{l}\right) p_{n/l}^{(k-1)/l}\right)\right]$$

$$\tilde{p}_n = \sum_{k=0}^{n-1} \tilde{p}_n^k.$$

First values

n	$\tilde{p}_n$
2	1
3	2
4	4
5	9
6	27
7	94
8	364
9	1529
10	6689
11	30230
12	140114

Sommaire

Branched covering of the sphere

Branched coverings of the sphere can be represented by bipartite planar maps having two properties:

Branched covering of the sphere

Branched coverings of the sphere can be represented by bipartite planar maps having two properties:

- (global) there are as many black vertices as faces.

Branched covering of the sphere

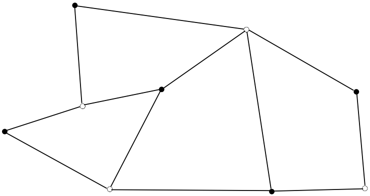
Branched coverings of the sphere can be represented by bipartite planar maps having two properties:

- (global) there are as many black vertices as faces.
- (local) for any (strict) subset $\mathcal{F}$ of faces of the map, the number of black vertex belonging to at least one face of $\mathcal{F}$ is strictly greater than the number of face in $\mathcal{F}$.

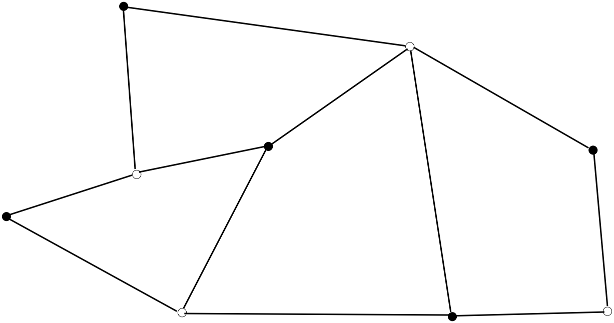
Branched covering of the sphere

Branched coverings of the sphere can be represented by bipartite planar maps having two properties:

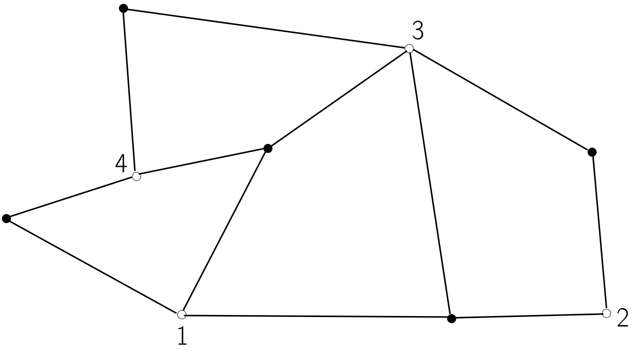
- (global) there are as many black vertices as faces.
- (local) for any (strict) subset $\mathcal{F}$ of faces of the map, the number of black vertex belonging to at least one face of $\mathcal{F}$ is strictly greater than the number of face in $\mathcal{F}$.



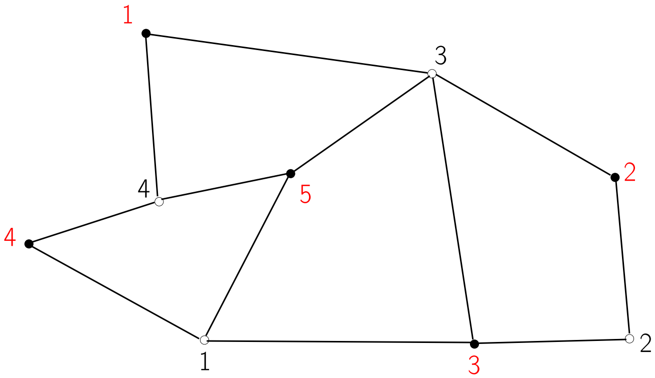
Hurwitz problem



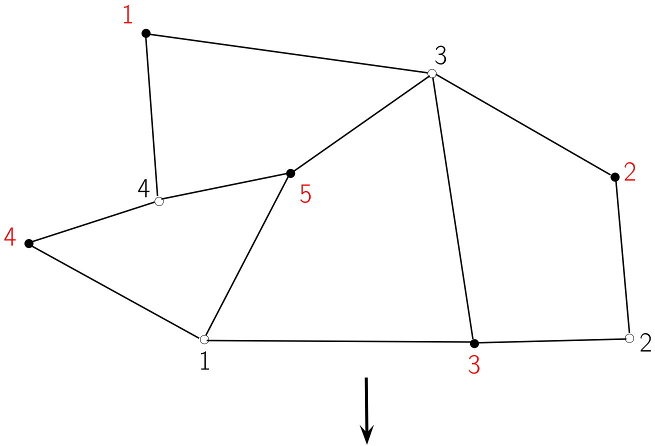
Hurwitz problem



Hurwitz problem



Hurwitz problem



$$(354)(23)(1532)(145) = 1$$

Thanks!