The escaping set of entire functions

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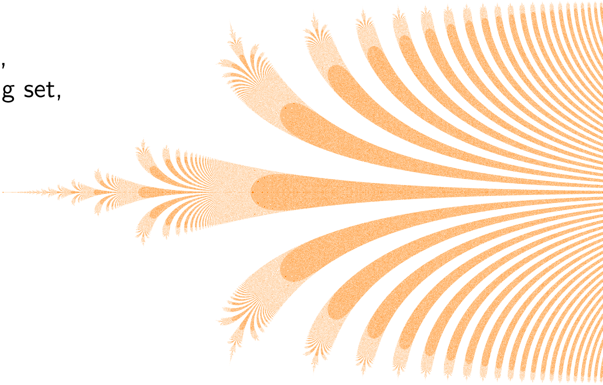
Sketch of the talk

1. The escaping set,
   Eremenko's conjecture,
   existence of hairs,
   and Cantor bouquets

2. The fast escaping set,
   the quite fast escaping set,
   slow escaping points,
   and spider's webs

3. Annular itineraries



The escaping set

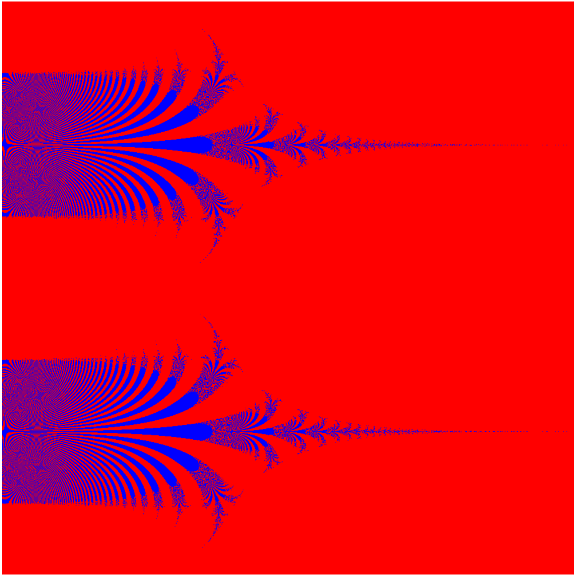
Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental function, the **escaping set** is

$$\mathcal{I}(f) := \{z \in \mathbb{C} \ : \ f^n(z) \to \infty \text{ as } n \to +\infty\}.$$

There are examples where the escaping set is:

- a union of disjoint curves tending to infinity (**Cantor bouquet**);
- a parabolic domain at infinity (**Baker domain**);
- a **wandering domain**.

Example: $f(z) = z + 1 + e^{-z}$



Blue: a Cantor bouquet — Red: a Baker domain.

Relation between $\mathcal{J}(f)$ and $\mathcal{I}(f)$

Theorem (Eremenko 1989)

Let f be an entire transcendental function. Then,

- $\mathcal{I}(f) \cap \mathcal{J}(f) \neq \emptyset$;
- $\mathcal{J}(f) = \partial \mathcal{I}(f)$;
- *all the components of* $\overline{\mathcal{I}(f)}$ *are unbounded;*
- *if* $f \in \mathcal{B}$, *then* $\mathcal{I}(f) \subseteq \mathcal{J}(f)$.

Alexandre E. Eremenko, *On the iteration of entire functions*, Dynamical Systems and Ergodic Theory, Banach Center Publ. **23** (1989), 339–345.

Eremenko's conjecture

On 1926, Fatou observes that for some entire functions (e.g. $f(z) = r \sin z$ for $r \in \mathbb{R}$) the Julia set contains curves of points that escapte to infinity under iteration and he already remarks that:

> *Il serait intéressant de rechercher si cette propriété nâappartiendrait pas à des substitutions beaucoup plus générales.*

On 1989 Eremenko made the following conjecture for entire functions:

> *It is plausible that the set $\mathcal{I}(f)$ always has the following property: every point $z \in \mathcal{I}(f)$ can be joined with ∞ by a curve in $\mathcal{I}(f)$.*

Pierre Fatou, *Sur lâitération des fonctions transcendantes entières*, Acta Math. **47** (1926), 337–370.

Alexandre E. Eremenko, *On the iteration of entire functions*, Dynamical Systems and Ergodic Theory, Banach Center Publ. **23** (1989), 339–345.

Good news for a large class of maps inside $\mathcal{B}$

Theorem (Rottenfußer, Rückert, Rempe, Schleicher 2011)

Let $f \in \mathcal{B}$ be a function of finite order, or more generally a finite composition of such functions. Then every point $z \in \mathcal{I}(f)$ can be connected to ∞ by a curve γ such that $f^n_{|\gamma} \to \infty$ uniformly.

- Proved for a special case (finite type + technical conditions) by **Robert L. Devaney** and **Folkert Tangerman** in 1986.
- **Krzysztof Barański** proved it for the disjoint-type case.
- With **Núria Fagella** we have proven a similar result for holomorphic self-maps of $\mathbb{C}^*$ which have two essential sing. (*in preparation*).

Robert L. Devaney, and Folkert Tangerman, *Dynamics of entire functions near the essential singularity*, Ergodic Theory Dynam. Systems **6** (1986), no. 4, 489–503.

Krzysztof Barański, *Trees and hairs for some hyperbolic entire maps of finite order*, Math Z. **257** (2007), no. 1, 33–59.

Günter Rottenfußer, Johannes Rückert, Lasse Rempe, and Dierk Schleicher, *Dynamic rays of bounded-type entire functions*, Ann. of Math. (2) **173** (2011), no. 1, 77–125.

Cantor *bouquets*

Definition

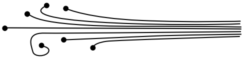
A **straight brush** is a subset B of $[0,\infty) \times (\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{R}^2$ such that

- $\forall (y,\alpha) \in B,\ \exists t_\alpha \in [0,\infty)$ such that $\{t\ :\ (t,\alpha) \in B\} = [t_\alpha, \infty)$;
- the set $\{\alpha\ :\ \exists y,\ (y,\alpha) \in B\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$ i $\forall (y,\alpha) \in B$, $\exists (\beta_n)_n, (\gamma_n)_n \subseteq \mathbb{R} \setminus \mathbb{Q},\ \beta_n \nearrow \alpha,\ \gamma_n \searrow \alpha$ such that $t_{\beta_n}, t_{\gamma_n} \to t_\alpha$;
- B is a closed subset of $\mathbb{R}^2$.

A **Cantor *bouquet*** is a set ambiently homeomorphic to a straight brush.

Topological properties:

- it has Hausdorff dimension 2, although it has Lebesgue measure 0;
- if Λ denotes the set of extreme points $e_\alpha = (t_\alpha, \alpha)$,
  Λ is a totally disconnected set,
  but $\Lambda \cup \{\infty\}$ is connected!



Jan M. Aarts and Lex G. Oversteegen, *The geometry of Julia sets*, Trans. Amer. Math. Soc. **338** (1993), 897-918.

Brushing the hairs of transcendental entire functions

Theorem (Barański, Jarque, Rempe 2012)

a) Let f be a disjoint-type function that can be written as a finite composition of finite-order functions in the class $\mathcal{B}$, then $\mathcal{J}(f)$ is a Cantor bouquet.
b) In general, if f is a finite order function in the class $\mathcal{B}$ (or, more generally, a composition of such functions) then the Julia set of such a function always contains a Cantor bouquet.

Krzysztof Barański, Xavier Jarque, and Lasse Rempe, *Brushing the hairs of transcendental entire functions*, Topology Appl. **159** (2012), no. 8, 2102-2114.

Maximum modulus

If f is an entire function and $R > 0$,

$$M(R, f) := \max_{|z|=R} |f(z)|.$$

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$$M(R, f) := \max_{|z|=R} |f(z)|.$$

⚠ We will usualy require that

$$R > 0 \text{ is such that } M(r, f) > r \text{ for } r \geqslant R.$$

Observe that this is equivalent to the fact that $M^n(R, f) \to \infty$ as $n \to \infty$, and it is enough to take

$$R > \min_{z \in \mathcal{J}(f)} |z|$$

by Montel's theorem.

The fast escaping set

Let f be an entire transcendental function, the **fast escaping set** is

$$\mathcal{A}(f) := \{z \in \mathbb{C} \ : \ \exists \ell \in \mathbb{N}, \ |f^{n+\ell}(z)| \geqslant M^n(R, f) \text{ for } n \in \mathbb{N}\},$$

where $R > 0$ is such that $M(r, f) > r$ for $r \geqslant R$.

Walter Bergweiler, and Aimo Hinkkanen, *On semiconjugation of entire functions*, Math. Proc. Cambridge Philos. Soc. **126** (1999) 565–574.

Philip J. Rippon, and Gwyneth M. Stallard, *Fast escaping points of entire functions*, Proc. London Math. Soc., **105** (2012), 787–820.

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where $R > 0$ is such that $M(r, f) > r$ for $r \geqslant R$.

Theorem (Bergweiler, Hinkkanen 1999)

Let f be an entire transcendental function. Then,

$$\mathcal{J}(f) = \partial \mathcal{A}(f).$$

Remember that Eremenko showed that $\mathcal{J}(f) = \partial \mathcal{I}(f)$.

Walter Bergweiler, and Aimo Hinkkanen, *On semiconjugation of entire functions*, Math. Proc. Cambridge Philos. Soc. **126** (1999) 565–574.

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Back to Eremenko's conjecture

Theorem (Rippon, Stallard 2005)

Let f be an entire transcendental function. All the components of $\mathcal{A}(f)$ are unbounded, and $\mathcal{A}(f) \neq \emptyset$. Hence, there is at least one unbounded component of $\mathcal{I}(f)$.

Philip J. Rippon, and Gwyneth M. Stallard, *On questions of Fatou and Eremenko*, Proc. Amer. Math. Soc. **133** (2005) 1119–1126.

Decomposition into level sets

$\mathcal{A}(f)$ can be decomposed into **level sets**:

$$A_R^{\ell}(f) := \{z \in \mathbb{C} \ : \ |f^n(z)| \geqslant M^{n+\ell}(R,f) \text{ for } n \in \mathbb{N}, \ n+\ell \in \mathbb{N}\};$$
$$A_R^{-\ell}(f) := \{z \in \mathbb{C} \ : \ |f^{n+\ell}(z)| \geqslant M^{n}(R,f) \text{ for } n \in \mathbb{N}, \ n+\ell \in \mathbb{N}\};$$
$$A_R(f) := A_R^0(f) := \{z \in \mathbb{C} \ : \ |f^n(z)| \geqslant M^{n}(R,f) \text{ for } n \in \mathbb{N}\}$$

are *closed* sets and

$$\mathcal{A}(f) = \bigcup_{\ell \in \mathbb{N}} A_R^{-\ell}(f) = \bigcup_{\ell \in \mathbb{N}} f^{-\ell}\big(A_R(f)\big).$$

Observe that $f(A_R^{\ell}(f)) \subseteq A_R^{\ell+1}(f) \subseteq A_R^{\ell}(f)$ for $\ell \in \mathbb{Z}$ and hence

$$A_R^{-\ell}(f) \subseteq A_R^{-(\ell+1)}(f) \text{ for } \ell \in \mathbb{N}.$$

Boundaries of escaping Fatou components

Theorem (Rippon, Stallard 2011)

Let f be a transcendental entire function, let $R > 0$ be such that $M(r, f) > r$ for $r \geqslant R$, and let $\ell \in \mathbb{Z}$. If U is a Fatou component that meets $A_R^{\ell}(f)$, then

(a) $\overline{U} \subseteq A_R^{\ell-1}(f)$;

(b) *if, in addition, U is simply connected, then* $\overline{U} \subseteq A_R^{\ell}(f)$.

In particular, if U is a Fatou component in $\mathcal{A}(f)$, then $\partial U \subseteq \mathcal{A}(f)$.

This is not true for $\mathcal{I}(f)$! In the example $f(z) = z + 1 + e^{-z}$, $\mathcal{F}(f)$ is connected (a Baker domain) and not all the points in $\partial\mathcal{F}(f) = \mathcal{J}(f)$ escape, there are periodic points. Baker domains are contained in the **slow escaping set**.

Philip J. Rippon, and Gwyneth M. Stallard, *Boundaries of escaping Fatou components*, Proc. Amer. Math. Soc. **139** (2011) 2807–2820.

The quite fast escaping set

The ***quite*** **fast escaping set** of an entire transc. function f is

$$\mathcal{Q}(f) := \{z \in \mathbb{C} \ : \ \exists \ell \in \mathbb{N}, \ |f^{n+\ell}(z)| \geqslant \mu_\varepsilon^n(R, f) \text{ for } n \in \mathbb{N}\},$$

where $\mu_\varepsilon(R, f) = M(R, f)^\varepsilon$, $0 < \varepsilon < 1$, and $\mu_\varepsilon^n(R, f) \to \infty$.

Philip J. Rippon, and Gwyneth M. Stallard, *Regularity and fast escaping points of entire functions*, preprint, 2013. arXiv:1301.2193v1 [math.DS].

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$$\mathcal{A}(f) \subseteq \mathcal{Q}(f) \subseteq \mathcal{I}(f).$$

Q: Are these sets always different?

Philip J. Rippon, and Gwyneth M. Stallard, *Regularity and fast escaping points of entire functions*, preprint, 2013. arXiv:1301.2193v1 [math.DS].

Slow escaping points

Theorem (Rippon, Stallard 2011)

Let f be a transcendental entire function. Then

$$\mathcal{A}(f) \neq \mathcal{I}(f), \quad \mathcal{Q}(f) \neq \mathcal{I}(f)$$

since $\mathcal{I}(f)$ cointains points that escape arbitrarily slowly:
given any positive sequence $(a_n)_n$ with $a_n \to \infty$ as $n \to +\infty$, there exists

$$\zeta \in \mathcal{I}(f) \cap \mathcal{J}(f), \text{ and } N \in \mathbb{N}$$

such that $|f^n(\zeta)| \leqslant a_n$ for $n \geqslant N$.

Philip J. Rippon, and Gwyneth M. Stallard, *Regularity and fast escaping points of entire functions*, preprint, 2013. arXiv:1301.2193v1 [math.DS].

The spider's web structure

We say that a set E is an (infinite) **spider's web** if E is connected and there exists a sequence of bounded simply connected domains G_n with $G_n \subseteq G_{n+1}$ for $n \in \mathbb{N}$, $\partial G_n \subseteq E$ for $n \in \mathbb{N}$ and

$$\bigcup_{n \geqslant 0} G_n = \mathbb{C}.$$

Theorem (Rippon, Stallard 2012)

Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geqslant R$. If $\mathbb{C} \setminus A_R(f)$ has a bounded component, then each of $A_R(f)$, $\mathcal{A}(f)$ and $\mathcal{I}(f)$ is a spider's web.

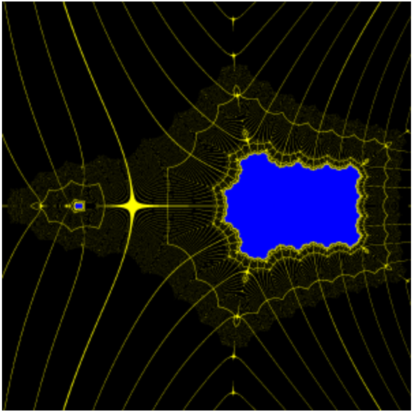
This cannot happen in class $\mathcal{B}$!

Philip J. Rippon, and Gwyneth M. Stallard, *Fast escaping points of entire functions*, Proc. London Math. Soc. **105** (2012) 787–820.

John W. Osborne, *The structure of spider's web fast escaping sets*, Bull. London Math. Soc. **44** (2012), 503–519.

Example of an spider's web

$$f(z) = \frac{1}{2}\left(\cos z^{1/4} + \cosh z^{1/4}\right)$$



(Picture borrowed from Dr. Dominique Fleischmann)

Annular itineraries

Let f be an entire transcendental function and let $R > 0$ be such that $M(r, f) > r$ for $r \geqslant R$. Define the sets $A_0(R) := \mathbb{D}_R$ and, for $n \geqslant 1$,

$$A_n(R) := \{z \in \mathbb{C} \; : \; M^{n-1}(R, f) \leqslant |z| < M^n(R, f)\},$$

which form a *partition* of the plane.

Philip J. Rippon, and Gwyneth M. Stallard, *Annular itineraries for entire functions*, preprint, 2013. arXiv:1301.1328v1 [math.DS]

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which form a *partition* of the plane.

The **annular itinerary** of z is $s_0 s_1 s_2 \ldots$, where

$$f^n(z) \in A_{s_n}(R),\; \text{for } n \geqslant 0.$$

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Maximum principle $\Rightarrow \ \forall n \in \mathbb{N}, \ s_{n+1} \leqslant s_n + 1.$

Philip J. Rippon, and Gwyneth M. Stallard, *Annular itineraries for entire functions*, preprint, 2013. arXiv:1301.1328v1 [math.DS]

Constructing orbits

Lemma

Let C_m, $m \geqslant 0$, be compact sets in $\mathbb{C}$ and f continuous such that

$$f(C_m) \supseteq C_{m+1}, \text{ for } m \geqslant 0.$$

Then $\exists \xi$ such that $f^m(\xi) \in C_m$, for $m \geqslant 0$.

Theorem (Rippon, Stallard 2013)

Let f be a transcendental entire function. There exists R and closed annuli

$$B_n := \{z \in \mathbb{C} \ : \ r_n \leqslant |z| < r_n'\}, \ n \in \mathbb{N},$$

all meeting $\mathcal{J}(f)$ such that

(a) $f(B_n) \supseteq B_{n+1}$, *for* $n \in \mathbb{N}$;

(b) $B_n \subseteq A_n$, *for* $n \in \mathbb{N}$;

(c) $\exists n_j \to +\infty$ *such that* $f(B_{n_j}) \supseteq B_n$, *for* $1 \leqslant n \leqslant n_j$, *with at most one exception.*

Prescribing itineraries

Observe that if $f^n(\xi) \in B_n$ for all $n \in \mathbb{N}$, then $\xi \in \mathcal{A}(f)$.

Theorem (Rippon, Stallard 2013)

Let f be a transcendental entire function. There exists $s \in \mathbb{N}$ and $n_j \to +\infty$ such that any itinerary with the following properties is achievable:

- $s_n \geqslant s$, *for* $n \geqslant 0$;
- $s_{n+1} = \begin{cases} s_n + 1, & \textit{if } n \neq n_j; \\ s_{n_j+1} \textit{ or one of } \{s, \ldots, s_{n_j}\} \textit{ with at most one exception}; \end{cases}$

In particular, we can construct points with the following itineraries:

- periodic itineraries, in given consecutive fundamenta annuli;
- uncountably many bounded itineraries;
- uncountably many unbounded non-escaping itineraries;
- arbitrarily slowly escaping itineraries.

Prescribing itineraries

Theorem (Rippon, Stallard 2013)

Let f be a transcendental entire function. There exists $R_0(f) > 0$ such that for any $(a_m)_m$ with $a_m \geqslant R_0(f)$ and $a_{m+1} \leqslant M(a_m, f)$, for $m \in \mathbb{N}$, there exist $\zeta \in \mathcal{J}(f)$ and $m_j \to \infty$ such that

- $|f^m(\zeta)| \geqslant a_m$, *for* $m \in \mathbb{N}$;
- $|f^{m_j}(\zeta)| \leqslant M^3(a_{m_j}, f)$, *for* $j \in \mathbb{N}$.

An application of this construction

Theorem (Rippon, Stallard 2013)

Let f be a transcendental entire function, then

$$\mathcal{Q}(f) = \mathcal{A}(f) \quad \Leftrightarrow \quad f \text{ is weakly regular.}$$

To obtain $\zeta \in \mathcal{Q}(f) \setminus \mathcal{A}(f)$ take

$$a_m = \mu_\varepsilon^m(R, f), \; m \in \mathbb{N}.$$

In particular, if $f \in \mathcal{B}$, then $\mathcal{Q}(f) = \mathcal{A}(f)$.

Philip J. Rippon, and Gwyneth M. Stallard, *Regularity and fast escaping points of entire functions*, preprint, 2013. arXiv:1301.2193v1 [math.DS]

THE END

Thank you for your attention!