Parabolic surgery and its application to Newton's method

Khudoyor Mamayusupov

Jacobs University Bremen

Universitat de Barcelona, June 10, 2013

Introduction

Let f be a rational map of degree $d \geq 3$. $crit(f)$-set of critical points of f. Its post-critical set is

$$Post(f) = \overline{\bigcup_{n \geq 0} f^{\circ n}(crit(f))}$$

If the number of accumulation points of post-critical set of a map is finite then the map is called *geometrically finite*, so no Herman ring and no Siegel disks are there. In this talk all maps are geometrically finite rational maps. If g.f. rational map does not have parabolic periodic points then it is called sub-hyperbolic. It is known that, Julia set of g.f. rational map has measure zero.

Newton Maps

Definition: Let $p(z)$ be a polynomial of $deg \geq 3$, its Newton map is

$$N_p(z) := z - \frac{p(z)}{p'(z)}$$

Some properties of Newton map, clearly the roots of $p(z) = 0$ are fixed points of Newton map, moreover, ∞ is a repelling fixed point of Newton map and for each root of $p(\xi_i) = 0$ of multiplicity $m_i \geq 1$, $N_p'(\xi) = (m_i - 1)/m_i$. Simple roots are superattracting fixed points of Newton map.

Newton Maps

Definition: Let $p(z)$ be a polynomial of $deg \geq 3$, its Newton map is

$$N_p(z) := z - \frac{p(z)}{p'(z)}$$

Some properties of Newton map, clearly the roots of $p(z) = 0$ are fixed points of Newton map, moreover, ∞ is a repelling fixed point of Newton map and for each root of $p(\xi_i) = 0$ of multiplicity $m_i \geq 1$, $N'_p(\xi) = (m_i - 1)/m_i$. Simple roots are superattracting fixed points of Newton map.

Observe that for polynomials p and q we have the uniform limit

$$\lim_{n \to \infty} N_{p(z)(1+\frac{q(z)}{n})^n}(z) = z - \frac{p(z)}{p'(z) + p(z)q'(z)} = N_{p(z)\cdot Exp(q(z))}(z)$$

Theorem (Rational Newton Map)

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire map: polynomial or transcendental function. Its Newton map N_f is rational if and only if there are polynomials $p(z)$ and $q(z)$ such that f has the form $f(z) = p(z) \cdot Exp(q(z))$. In this case, ∞ is a repelling or parabolic fixed point.

Theorem (Rational Newton Map)

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire map: polynomial or transcendental function. Its Newton map N_f is rational if and only if there are polynomials $p(z)$ and $q(z)$ such that f has the form $f(z) = p(z) \cdot Exp(q(z))$. In this case, ∞ is a repelling or parabolic fixed point.

More precisely, let $m, n \geq 0$ be the degrees of p and q, respectively. If $m = 1$ and $n = 0$, then N_f is constant. If $m \geq 2$ and $n = 0$, then ∞ is repelling with multiplier $\frac{m}{m-1}$. If $n \geq 1$, then ∞ is parabolic with multiplier $+1$ and multiplicity $n + 1 \geq 2$.

Theorem (Rational Newton Map)

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire map: polynomial or transcendental function. Its Newton map N_f is rational if and only if there are polynomials $p(z)$ and $q(z)$ such that f has the form $f(z) = p(z) \cdot Exp(q(z))$. In this case, ∞ is a repelling or parabolic fixed point.

More precisely, let $m, n \geq 0$ be the degrees of p and q, respectively. If $m = 1$ and $n = 0$, then N_f is constant. If $m \geq 2$ and $n = 0$, then ∞ is repelling with multiplier $\frac{m}{m-1}$. If $n \geq 1$, then ∞ is parabolic with multiplier $+1$ and multiplicity $n + 1 \geq 2$.

(Obviously, the roots of pe^q are those of p.) Rational Newton map has a form

$$N_{pe^q}(z) = z - \frac{p(z)}{p'(z) + p(z)q'(z)}$$

for polynomials p and q.

Theorem (Shishikura)

If a rational map has only one fixed point which is repelling or parabolic with multiplier 1, then its Julia set is connected. In particular, Julia set of a (rational) Newton map is connected.

Theorem (Shishikura)

If a rational map has only one fixed point which is repelling or parabolic with multiplier 1, then its Julia set is connected. In particular, Julia set of a (rational) Newton map is connected.

Definition (Immediate Basin)

Let f be a (rational) Newton map and $\xi \in \mathbb{C}$ be its fixed point. Let $A_\xi = \{z \in \mathbb{C} : \lim_{n\to\infty} f^{\circ n}(z) = \xi\}$ be the *basin (of attraction)* of ξ. The component of A_ξ containing ξ is called the *immediate basin* of ξ and denoted U_ξ.

Theorem (Shishikura)

If a rational map has only one fixed point which is repelling or parabolic with multiplier 1, then its Julia set is connected. In particular, Julia set of a (rational) Newton map is connected.

Definition (Immediate Basin)

Let f be a (rational) Newton map and $\xi \in \mathbb{C}$ be its fixed point. Let $A_\xi = \{z \in \mathbb{C} : \lim_{n\to\infty} f^{\circ n}(z) = \xi\}$ be the *basin (of attraction)* of ξ. The component of A_ξ containing ξ is called the *immediate basin* of ξ and denoted U_ξ.

Theorem (Immediate Basins Simply connected)

The immediate basin of a rational Newton map is simply connected and unbounded.

For a Newton map of $f = pe^q$, the area of every immediate basin is finite if $deg(q) \geq 3$ [Mako Haruta] and infinite if $p(z) = z$ and $deg(q) \in \{0, 1\}$ [Figen Cilinger]. Within immediate basin rational Newton maps can be easily characterized.

Theorem

Let N_f be a (rational) Newton map of $f = pe^q$ and $\xi \in \mathbb{C}$ is a root of p and $U = U_\xi$ be its immediate basin. Then U contains $k \geq 1$ critical points of N_f and $N_f|_U$ is a proper self-map of degree $k+1$.

If we use the fact that immediate basins are simply connected then we see that restricted maps $N_f|_U$ are conformal conjugate to Blaschke products: Let R be a Riemann map $R : U \to \mathbb{D}$ then $R \circ N_f|_U \circ R^{-1}$ is a Blaschke product.

Parabolic fixed points

Let $\xi \in \mathbb{C}$ be a fixed point of f. If $f'(\xi)$ is a k -th root of unity: $f'(\xi) = e^{2\pi i \frac{m}{k}}$ then

$$f^{\circ k}(z) = \xi + c(z-\xi)^{n+1} + O((z-\xi)^{n+2})$$

near ξ, $n+1$ is called multiplicity of ξ and $k|n$. There are n attracting and repelling directions alternating at ξ. In this talk multiplier of parabolic fixed point will be $+1$.

Parabolic fixed points

Let $\xi \in \mathbb{C}$ be a fixed point of f. If $f'(\xi)$ is a k -th root of unity: $f'(\xi) = e^{2\pi i \frac{m}{k}}$ then

$$f^{\circ k}(z) = \xi + c(z-\xi)^{n+1} + O((z-\xi)^{n+2})$$

near ξ, $n+1$ is called multiplicity of ξ and $k|n$. There are n attracting and repelling directions alternating at ξ. In this talk multiplier of parabolic fixed point will be $+1$.
We denote

$$A_\xi = \{z \in \mathbb{C} : \lim_{n\to\infty} f^{\circ n}(z) = \xi\}$$

the *basin (of attraction)* of parabolic fixed point ξ. By definition the immediate basin is the unique forward invariant connected component of A_ξ. Thus, there are n immediate basins denoted by U_j, $j \in \{1..n\}$.

Surgery method

Theorem (P. Haissinsky)

Let f be a sub-hyperbolic rational map of degree $\deg(f) \geq 2$ with connected Julia set and with an attracting fixed point α and a repelling fixed point $\beta \in \partial U_\alpha$ and accessible from the basin U_α. Then there exist a rational map g of the same degree as f has and a David homeomorphism $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, such that:

1. *$\phi \circ f = g \circ \phi$ on $\overline{\mathbb{C}} \setminus U_\alpha$; in particular $\phi : J_f \to J_g$ is a homeomorphism which conjugates f and g on the Julia sets.*
2. *ϕ is conformal on $\overline{\mathbb{C}} \setminus \overline{A_\alpha}$-in the complement of basin of α*
3. *$\beta' = \phi(\beta)$ is a parabolic fixed point with multiplier $+1$ and $\phi(U_\alpha)$ is one of the immediate parabolic basins of β'*

Surgery method

Theorem (P. Haissinsky)

Let f be a sub-hyperbolic rational map of degree $\deg(f) \geq 2$ with connected Julia set and with an attracting fixed point α and a repelling fixed point $\beta \in \partial U_\alpha$ and accessible from the basin U_α. Then there exist a rational map g of the same degree as f has and a David homeomorphism $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, such that:

1. $\phi \circ f = g \circ \phi$ *on* $\overline{\mathbb{C}} \setminus U_\alpha$; *in particular* $\phi : J_f \to J_g$ *is a homeomorphism which conjugates* f *and* g *on the Julia sets.*
2. ϕ *is conformal on* $\overline{\mathbb{C}} \setminus \overline{A_\alpha}$*-in the complement of basin of* α
3. $\beta' = \phi(\beta)$ *is a parabolic fixed point with multiplier* $+1$ *and* $\phi(U_\alpha)$ *is one of the immediate parabolic basins of* β'

Remark 1. If you start with attracting periodic point with k period and β is a repelling periodic point on the boundary with k' period and $k'|k$ then theorem holds true. In this case $\beta' = \phi(\beta)$ is a k' periodic parabolic point with multiplier k/k' root of unity.

Application of surgery to Newton maps

Remark 2. You can also take finite number of basins with repelling fixed point at the common boundary and simultaneously apply the surgery. This is crucial fact for changing basins of Newton maps. Resulting map will be a rational map which satisfies all properties of being a rational Newton map. Since it has one parabolic fixed point, so, it is a Newton map of pe^{q} for some polynomials p and q. We have the following.

Application of surgery to Newton maps

Remark 2. You can also take finite number of basins with repelling fixed point at the common boundary and simultaneously apply the surgery. This is crucial fact for changing basins of Newton maps. Resulting map will be a rational map which satisfies all properties of being a rational Newton map. Since it has one parabolic fixed point, so, it is a Newton map of pe^q for some polynomials p and q. We have the following.

Theorem

Let N_p be a sub-hyperbolic Newton map of polynomial p. Let the number of different roots of p be $m+n \geq 3$. Note that also, $deg(N_p) = m+n$. Take $n \geq 1$ roots $\xi_j, j \in \{1..n\}$ and immediate basins $U_{\xi_j}, j \in \{1..n\}$. Then there is a David homeomorphism ϕ and polynomials $\tilde{p}$ and $\tilde{q}$ of degree m,n, respectively, such that N_p and $N_{\tilde{p}e^{\tilde{q}}}$ are conformal conjugate on $\overline{\mathbb{C}} \setminus \bigcup_{1 \leq j \leq n} \overline{A_{\xi_j}}$: in particular they are conjugate on Julia sets.

David-Beltrami equation

We consider Beltrami differential (form) $\mu\frac{d\bar{z}}{dz}$ for which there exists homeomorphic solutions $\phi : U \to U'$ of the Beltrami equation

$$\bar{\partial}\phi = \mu\partial\phi$$

with $\phi \in W^{1,r}_{loc}, r \geq 1$, where U, U' are domains of $\mathbb{C}$. $W^{1,r}_{loc}$ is the Sobolev space of $L^1_{loc}(U)$ functions with $L^r_{loc}(U)$ distributional derivatives $\bar{\partial}\phi, \partial\phi$. Let $w = f(z)$ be C^1 homeomorphism then differential is

$$df = \partial f dz + \bar{\partial} f d\bar{z}$$

and complex dilatation of f is $K_f = \frac{1+|\mu_f|}{1-|\mu_f|}$, where $\mu_f = \frac{\bar{\partial}f}{\partial f}\frac{d\bar{z}}{dz}$ -Beltrami form of f on domain U.

Definition

Beltrami differential $\mu = \mu(z)\frac{d\bar{z}}{dz}$ is called David-Beltrami differential if there exist constants $M > 0, \alpha > 0$ *and* $K_0 > 1$ such that

$$\forall K > K_0 : Area(\{z \in U | K_\mu(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|} > K\}) \leq Me^{-\alpha K}$$

Theorem (David)

Any Beltrami-David equation has a solution $\phi : U \to U'$ o-p homeomorphism in $L^1_{loc}(U)$ such that:

- *ϕ belongs to $L^r_{loc}(U)$ for all $0 < r < 2$*
- *ϕ is unique by post composition by a conformal map.*
- *ϕ is absolutely continuous: For a measurable set $E \subset U$*

$$area(E) = 0 \Longleftrightarrow area(\phi(E)) = 0$$

Such a map is called David homeomorphism.

Haissinsky surgery. Local dynamics of model maps.

Let us sketch a proof of Haissinsky theorem. We consider the map $f : z \to \lambda z$ with $\lambda z > 1$ (repelling model map) and sector $S = \{z \in \mathbb{C}\, |Arg(z)| < \theta \, and \, 0 < |z| < 1\}$.
The mapping

$$z \to w = \frac{log \lambda}{Log(z)}$$

conjugates f to $g : w \to \frac{w}{w+1}$ (parabolic model map) defined at the cusp $Cusp = w(S)$ at the origin. Denote by Q_n^f-the quadrilateral bounded by segments $[\frac{1}{\lambda^{n+1}} e^{\pm i\theta}, \frac{1}{\lambda^n} e^{\pm i\theta}]$ and arcs of radius $\frac{1}{\lambda^{n+1}}$ and $\frac{1}{\lambda^n}$ contained in $\mathbb{D} \setminus S$.

Haissinsky surgery. Local dynamics of model maps.

Let us sketch a proof of Haissinsky theorem. We consider the map $f: z \to \lambda z$ with $\lambda z > 1$ (repelling model map) and sector $S = \{z \in \mathbb{C}\, |Arg(z)| < \theta \, and \, 0 < |z| < 1\}$.
The mapping

$$z \to w = \frac{log \lambda}{Log(z)}$$

conjugates f to $g: w \to \frac{w}{w+1}$ (parabolic model map) defined at the cusp $Cusp = w(S)$ at the origin. Denote by Q_n^f-the quadrilateral bounded by segments $[\frac{1}{\lambda^{n+1}} e^{\pm i\theta}, \frac{1}{\lambda^n} e^{\pm i\theta}]$ and arcs of radius $\frac{1}{\lambda^{n+1}}$ and $\frac{1}{\lambda^n}$ contained in $\mathbb{D} \setminus S$.

Lemma

There exists extension $z \to w(z)$ to a neighborhood of the origin a piecewise C^1 homeomorphism χ, such that $K_\chi \approx n$ on Q_n^f

Here for positive numbers $a \approx b$ means there exists positive constant c such that $\frac{1}{c} < \frac{a}{b} < c$.

Proof of lemma.

It is easy to check the following map is an extension we need. Define

$$\chi : \mathbb{D} \setminus \overline{S} \to \mathbb{D} \setminus \overline{Cusp}$$

by

$$\chi :\ \rho e^{it} \to \frac{log\lambda}{|log(\rho) + i\theta|} \cdot e^{ia_\rho(t)}$$

where $a_\rho(t)$ is an affine map and it can be computed easily making the extension continuous.

Proof of lemma.

It is easy to check the following map is an extension we need. Define

$$\chi : \mathbb{D} \setminus \overline{S} \to \mathbb{D} \setminus \overline{Cusp}$$

by

$$\chi : \ \rho e^{it} \to \frac{log \lambda}{|log(\rho) + i\theta|} \cdot e^{ia_\rho(t)}$$

where $a_\rho(t)$ is an affine map and it can be computed easily making the extension continuous.
χ takes arcs of circles to arcs of circles. Let $Q_n^g = \chi(Q_n^f)$, since $mod(Q_n^f) \approx 1$ and $mod(Q_n^g) \approx log(1 + \frac{1}{n}) \approx \frac{1}{n}$ we get

$$K_\chi \approx \frac{mod(Q_n^f)}{mod(Q_n^g)} \approx n$$

Comparison of basins

As we know from above that, restriction map is conjugate to Blaschke product. We have to understand how Model maps are transforming into each other. Let

$$B(z) = \frac{z^d + b}{1 + bz^d} \textit{ and } B_{par}(z) = \frac{z^d + a}{1 + az^d}$$

be two Blaschke products, where $0 < b < a = \frac{d-1}{d+1}$. Both leave unit disc, unit circle and compliment of closed unit disc invariant. $J_B = J_{B_{par}} = \mathbb{S}^1$. 1 is repelling fixed point for B and parabolic for B_{par}, with $B'_{par}(1) = 1$. 0 and ∞ are the only critical points of both maps. B has a fixed point α in the interval $(0, 1)$, with real multiplier. Immediate basins are $U_\alpha = \mathbb{D}$ and for parabolic fixed point 1 of B_{par} we have $U_1 = \mathbb{D}$ and $U_2 = \mathbb{C} \setminus \overline{\mathbb{D}}$

Lemma (Blaschke surgery)

There exist a homeomorphism $\phi : \mathbb{D} \to \mathbb{D}$ piecewise C^1 and sector $S_B \subset \mathbb{D}$ which is a neighborhood of α with vertex at 1 s.t.

1. *for all $z \in \mathbb{D} \backslash S_B$ we have $\phi \circ B(z) = B_{par} \circ \phi(z)$*
2. *There is a set S'_B which is intersection of S_B with some neighborhood of 1, such that $\phi : \mathbb{D} \backslash \bigcup B^{-n}(S'_B) \to \phi(\mathbb{D} \backslash \bigcup B^{-n}(S'_B))$ is a quasiconformal map.*
3. *On quadrilaterals Q_n^B in S'_B defined as Q_n^f for repelling model map, we have $K_\phi \approx n$ for all n large*

Sketch of proof. Note that construction of conjugating map ϕ is essentially the same as model maps. If you are close to 1 then as we did for model maps we construct conjugacy between B and B_{par} in the complement of sector at 1. In that sector we just make the map to be smooth and homeomorphism as above lemma. Since Fatou map and linearizing maps are conformal, the dilatation of ϕ stays to be $\approx$ n.

Topological surgery

Let f be a sub-hyperbolic map with an attracting fixed point α with real multiplier $f'(\alpha)$ having only one critical point in the immediate basin $U = U_\alpha$. Let R be a Riemann map $R : U \to \mathbb{D}$ so that $R \circ f|_U \circ R^{-1}$ is a Blaschke product. Let ϕ be a homeomorphism defined in above Blaschke lemma, which partially conjugates B and B_{par}. Let $\hat{B} = \phi \circ B_{par} \circ \phi^{-1}$, note that $\hat{B} = B$ on $\mathbb{D} \backslash S_B$.
Let $H = R^{-1} \circ \hat{B} \circ R$ on U.

Topological surgery

Let f be a sub-hyperbolic map with an attracting fixed point α with real multiplier $f'(\alpha)$ having only one critical point in the immediate basin $U = U_\alpha$. Let R be a Riemann map $R : U \to \mathbb{D}$ so that $R \circ f|_U \circ R^{-1}$ is a Blaschke product. Let ϕ be a homeomorphism defined in above Blaschke lemma, which partially conjugates B and B_{par}. Let $\hat{B} = \phi \circ B_{par} \circ \phi^{-1}$, note that $\hat{B} = B$ on $\mathbb{D} \backslash S_B$.
Let $H = R^{-1} \circ \hat{B} \circ R$ on U.
Let us define topological model map $G = \begin{cases} \mathrm{H(z)}, & \text{if } z \in U; \\ \mathrm{f(z)}, & \text{elsewhere.} \end{cases}$
We want to define G -invariant complex structure. Let $\hat{\mu} = \bar{\partial}\phi / \partial\phi$, then it is invariant by $\hat{B}$. Naturally we pull it back to the immediate basin by $R : \mu = R^* \hat{\mu}$. We spread it by dynamics and define almost complex structure by $\mu = \begin{cases} (\mathrm{f}^n)^* \mu, & \text{on } f^{-n}(U); \\ 0, & \text{otherwise.} \end{cases}$

μ is Beltrami-David form

By construction μ is G-invariant.

Lemma

If $\beta \notin Post(f)$ then μ is a Beltrami-David form.

μ is Beltrami-David form

By construction μ is G-invariant.

Lemma

If $\beta \notin Post(f)$ then μ is a Beltrami-David form.

Proof There exist linearizable neighborhood V of 1 disjoint from $Post(f)$. Note that $f^{-1}(V) \subset\subset V$, let $\Sigma_\beta = f^{-1}(V) \cap R^{-1}(S'_B)$. Set $\rho = |f'(\beta)|$ and K_μ dilatation of μ. Using Blaschke surgery lemma and Koebe distortion theorem we obtain

$$Area(\{z \in \Sigma_\beta | K_\mu > n\}) \approx (1/\rho^{2n}) Area(\Sigma_\beta)$$

Let $X = \cup_{f^k(y)=\beta} \Sigma_y$ -union of all domains under all preimages of Σ_β under map f. Using the fact that $f^k : (f^{-k}(V), y) \to (V, \beta)$ is conformal we get following estimate

cont..

$Area(\{z \in \hat{\mathbb{C}} : K_\mu > n\}) = \sum_{k \geq 0} \sum_{f^k(y)=\beta} Area(z \in \Sigma_y : K_\mu > n)$
$\leq (C\rho^{-2n}) Area(X)$,
for some constant C. Since we are working in sphere with spherical metric, the area of X is finite. This ends the proof of lemma.

cont..

$Area(\{z \in \hat{\mathbb{C}} : K_\mu > n\}) = \sum_{k \geq 0} \sum_{f^k(y)=\beta} Area(z \in \Sigma_y : K_\mu > n)$
$\leq (C\rho^{-2n}) Area(X)$,
for some constant C. Since we are working in sphere with spherical metric, the area of X is finite. This ends the proof of lemma.

Finishing the proof of Haissinsky theorem

cont..

$Area(\{z \in \hat{\mathbb{C}} : K_\mu > n\}) = \sum_{k \geq 0} \sum_{f^k(y)=\beta} Area(z \in \Sigma_y : K_\mu > n)$
$\leq (C\rho^{-2n})Area(X)$,
for some constant C. Since we are working in sphere with spherical metric, the area of X is finite. This ends the proof of lemma.

Finishing the proof of Haissinsky theorem

Since ϕ satisfies Beltrami-David differential equation, when G is injective we need to check that $\phi \circ G \in W^{1,1}_{loc}(\overline{\mathbb{C}} \backslash \partial U)$ and is also solution of the same Beltrami-David equation.

Thank you !!!