From quasiconformal foldings to entire functions

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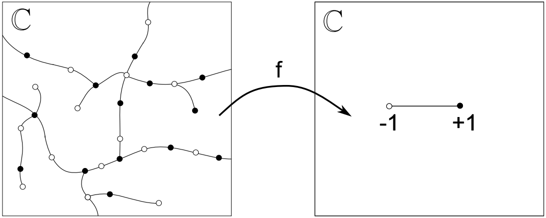
Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function with

- no finite asymptotic values
- exactly two critical values, say $\{-1, +1\}$

Question: What does f "look like" ??

$T = f^{-1}([-1, +1])$ is an infinite bipartite tree.



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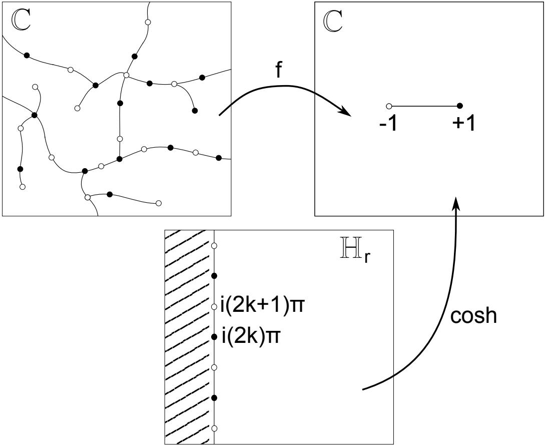

$\cosh : \mathbb{H}_r \to \mathbb{C} \backslash [-1, +1]$ is a universal cover.

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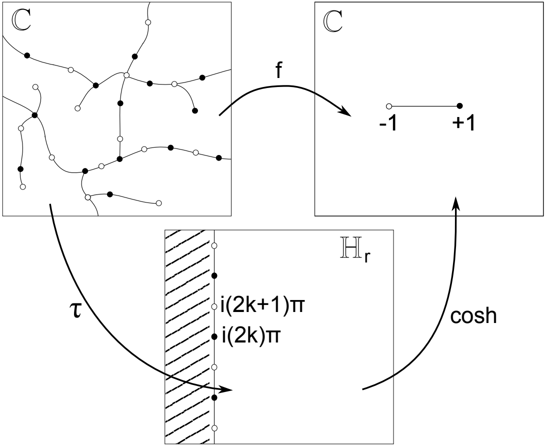

$\forall\Omega$ c.c. of $\mathbb{C}\setminus T$, $\tau_{|\Omega} = (\cosh^{-1} \circ f_{|\Omega}) : \Omega \to \mathbb{H}_r$ is conformal.

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More precisely, given

- an infinite bipartite tree $T \subset \mathbb{C}$ with "smooth" enough geometry
- a map τ such that $\tau_{|\Omega} : \Omega \to \mathbb{H}_r$ is conformal, $\forall \Omega$ c.c. of $\mathbb{C} \setminus T$

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Main problem: τ is not continuous across T in general.

Solution: Replace (T, τ) by (T', η) such that

- $T \subset T'$
- $\eta_{|\Omega'} : \Omega' \to \mathbb{H}_r$ is quasiconformal, $\forall \Omega'$ c.c. of $\mathbb{C} \setminus T'$
- $\eta = \tau$ off a small neighborhood of T'
- $\cosh \circ \eta$ is continuous across T'

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Then apply Ahlfors-Bers theorem:

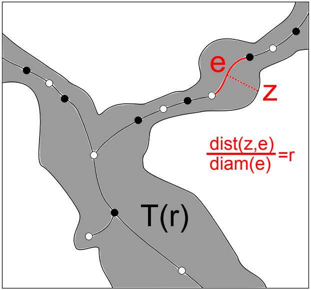
$\exists$ an entire function f and a quasiconformal map ϕ such that

$$f \circ \phi = \cosh \circ \tau \text{ off a small neighborhood of } T$$

The neighborhood of T

For every $r > 0$, define an open neighborhood of T as follows

$$T(r) = \bigcup_{e \text{ edge of } T} \left\{ z \in \mathbb{C} \text{ such that } \mathrm{dist}(z, e) < r\mathrm{diam}(e) \right\}$$



Lemma 0

If T has bounded geometry, namely

1. edges of T are $\mathcal{C}^2$ with uniform bounds
2. angles between adjacent edges are uniformly bounded away from 0
3. adjacent edges have uniformly comparable length
4. for non-adjacent edges e and f, $\frac{\mathrm{diam}(e)}{\mathrm{dist}(e,f)}$ is uniformly bounded

then there exists $r_0 > 0$ such that

$$\forall \Omega \text{ c.c. of } \mathbb{C} \setminus T, \text{ and } \forall \text{ edge } e \subset \partial\Omega,$$
$$\text{the square in } \mathbb{H}_r \text{ that has } \tau_{|\Omega}(e) \text{ as one side is in } \tau_{|\Omega}\Big(T(r_0) \cap \Omega\Big)$$

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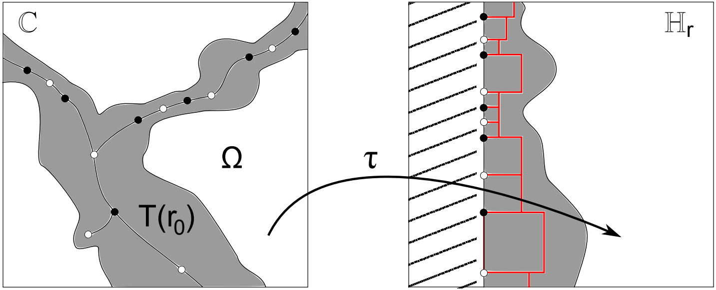


Theorem 1 (Bishop 2011)

If (T, τ) satisfies the following conditions

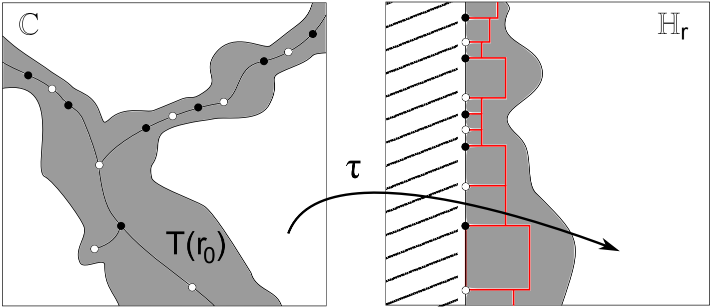
1. T has bounded geometry
2. every edge has τ-size $\geqslant \pi$

then there exist an entire function f and a quasiconformal map ϕ such that

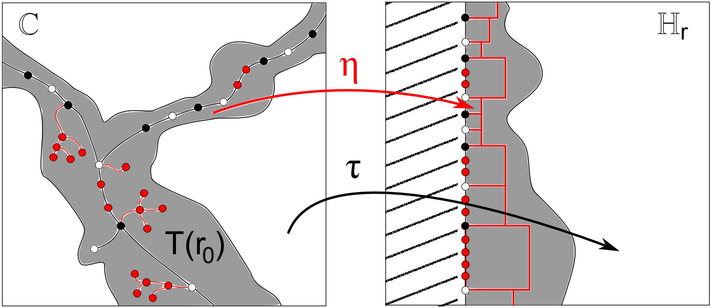
$$f \circ \phi = \cosh \circ \tau \text{ off } T(r_0)$$

Moreover

$$\begin{cases} f \text{ has no asymptotic values} \\ \text{the only critical values of } f \text{ are } \{-1, +1\} \\ \phi(T) \subset f^{-1}([-1, +1]) \quad (= \phi(T')) \end{cases}$$



$\mathbb{C}$
$T(r_0)$
τ
$\mathbb{H}_r$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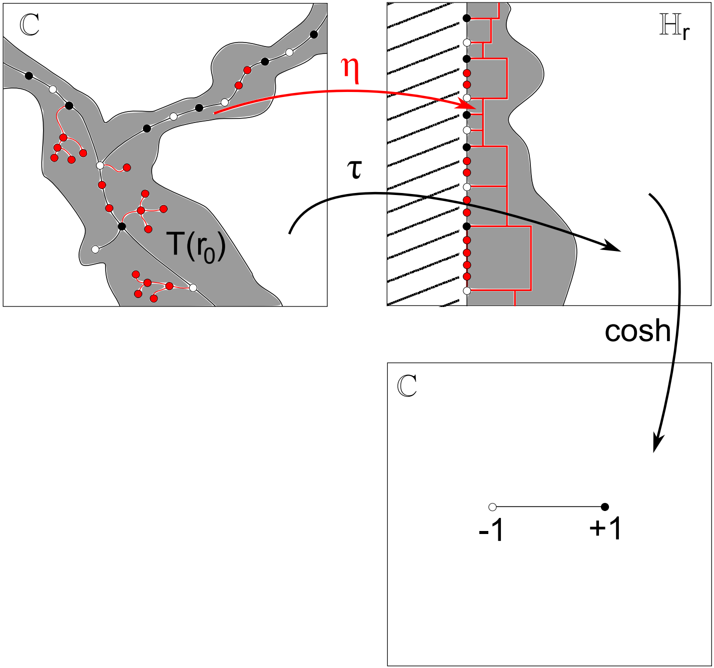

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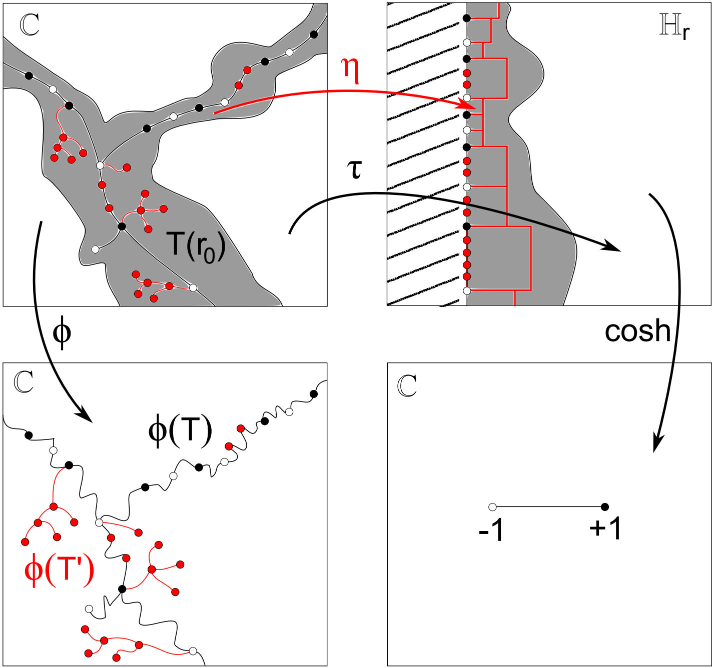

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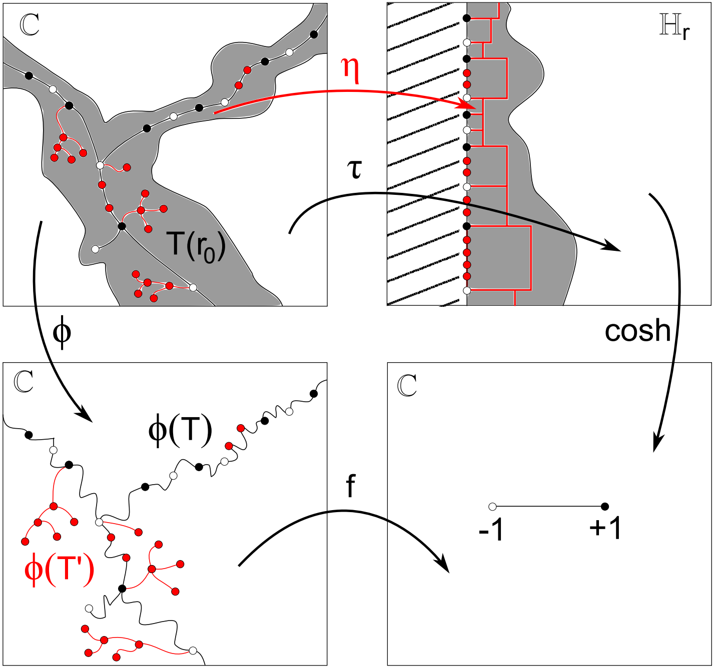

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Sketch of the proof: Construct (T', η) such that

- $T \subset T' \subset T(r_0)$
- $\eta_{|\Omega'} : \Omega' \to \mathbb{H}_r$ is quasiconformal, $\forall \Omega'$ c.c. of $\mathbb{C} \setminus T'$
- $\eta = \tau$ off $T(r_0)$
- $\cosh \circ \eta$ is continuous across T'
- $T' = (\cosh \circ \eta)^{-1}([-1, +1])$

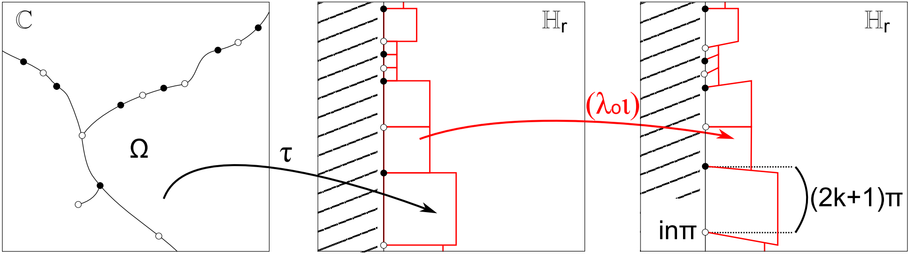
Particular case: $\forall$ edge $e \subset \partial\Omega \cup \partial\Omega'$, $\operatorname{diam}(\tau_{|\Omega}(e)) = \operatorname{diam}(\tau_{|\Omega'}(e)) \geqslant \pi$

Lemma 1

There exists $K_1 \geqslant 1$ such that
$\forall \Omega$ c.c. of $\mathbb{C} \setminus T$, $\exists$ a K_1-quasiconformal map $(\lambda_\Omega \circ \imath_\Omega) : \mathbb{H}_r \to \mathbb{H}_r$ /

- $(\lambda_\Omega \circ \imath_\Omega) = \mathrm{Id}$ off $\tau_{|\Omega}\Big(T(r_0) \cap \Omega\Big)$
- $\forall$ edge $e \subset \partial\Omega$, $(\lambda_\Omega \circ \imath_\Omega)\Big(\tau_{|\Omega}(e)\Big) = i\Big[n\pi, n\pi + (2k+1)\pi\Big]$
- $\forall$ edge $e \subset \partial\Omega \cup \partial\Omega'$, $(\lambda_\Omega \circ \imath_\Omega) \circ \tau_{|\Omega} = (\lambda_{\Omega'} \circ \imath_{\Omega'}) \circ \tau_{|\Omega'} + im\pi$ on e

$$\begin{cases} \imath_\Omega : \mathbb{H}_r \to \mathbb{H}_r \text{ moves the vertices into } i\mathbb{Z}\pi \\ \lambda_\Omega : \mathbb{H}_r \to \mathbb{H}_r \text{ fixes } i\mathbb{Z}\pi \text{ and makes the continuity across } T \end{cases}$$



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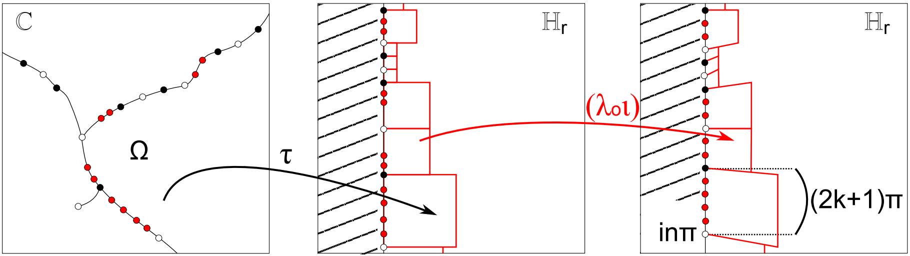

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Then define

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General case: $\forall$ edge $e \subset \partial\Omega \cup \partial\Omega'$, $\min\{\mathrm{diam}(\tau_{|\Omega}(e)), \mathrm{diam}(\tau_{|\Omega'}(e))\} \geqslant \pi$

We may assume $\tau_{|\Omega}(e) = i\Big[n\pi, n\pi + (2k+1)\pi\Big]$, $\forall$ edge $e \subset \partial\Omega$.

Lemma 2 (quasiconformal folding)

There exists $K_2 \geqslant 1$ such that

$\forall\Omega$ c.c. of $\mathbb{C}\backslash T$, $\exists$ a K_2-quasiconformal map $\psi_\Omega : W_\Omega \subset \mathbb{H}_r \to \mathbb{H}_r$ /

- $\psi_\Omega = \mathrm{Id}$ off $\tau_{|\Omega}\Big(T(r_0) \cap \Omega\Big)$
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Exercise: Find a quasiconformal map ψ from a square to itself such that

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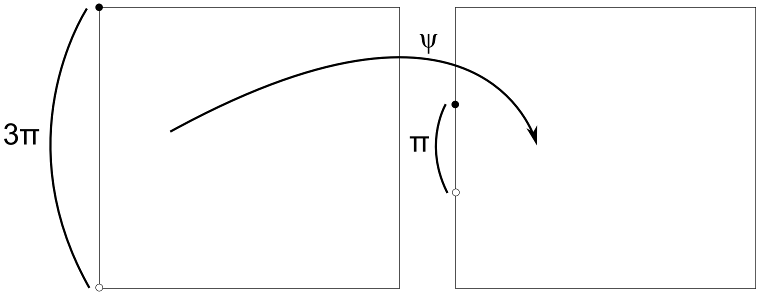

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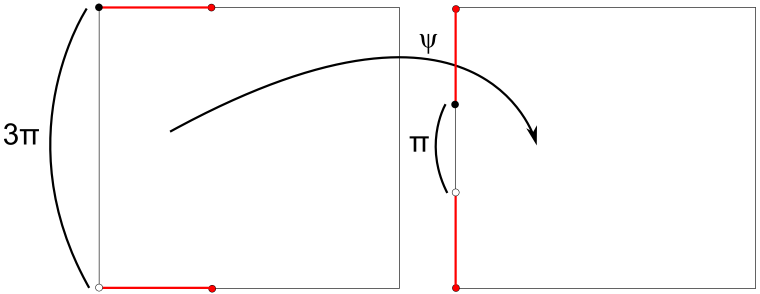

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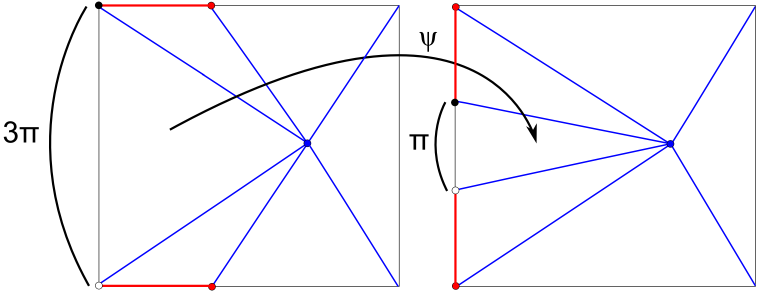

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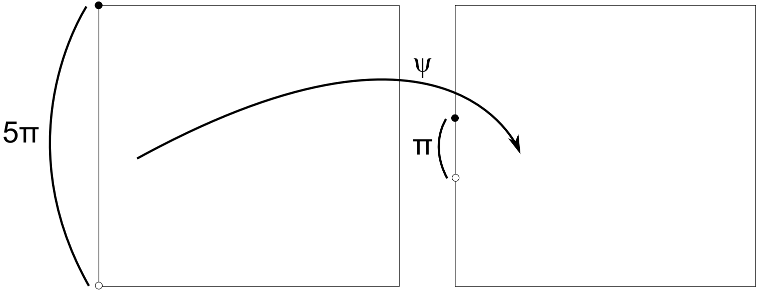

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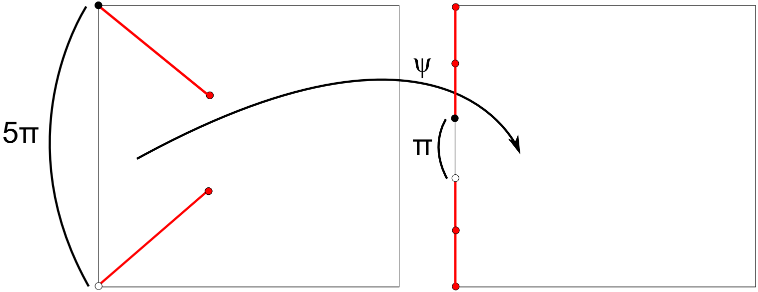

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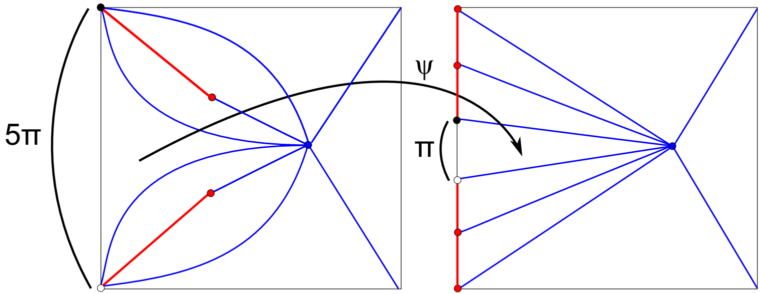

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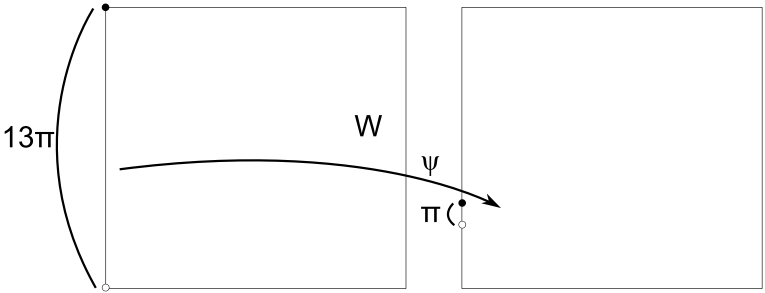

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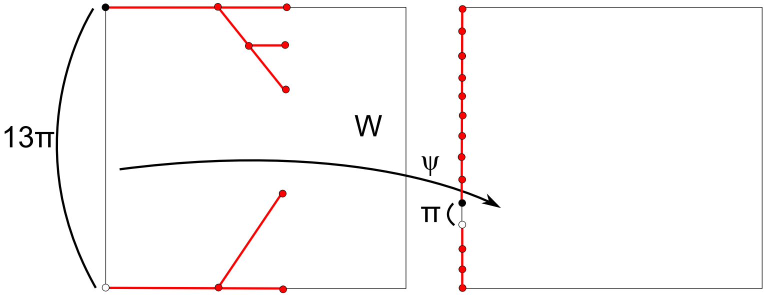

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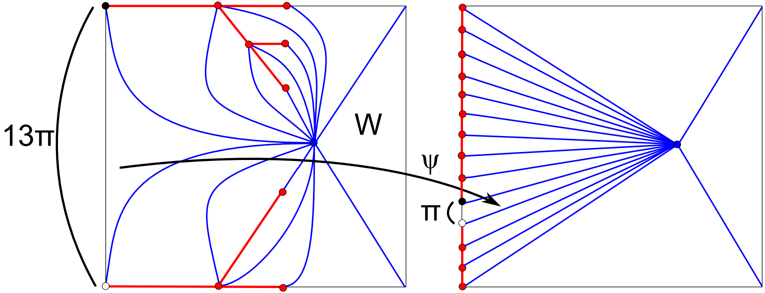

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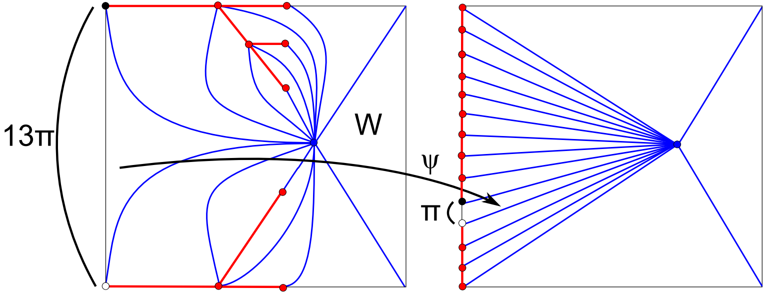

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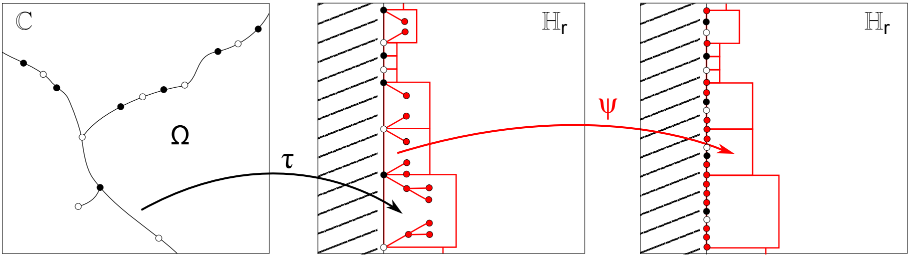
Claim of Lemma 2: The dilatation of ψ is uniformly bounded for every side length of square.

$\psi_\Omega : \mathbb{H}_r \to \mathbb{H}_r$ realizes an unfolding in each square of $\tau_{|\Omega}\Big(T(r_0) \cap \Omega\Big)$ and makes the continuity across T



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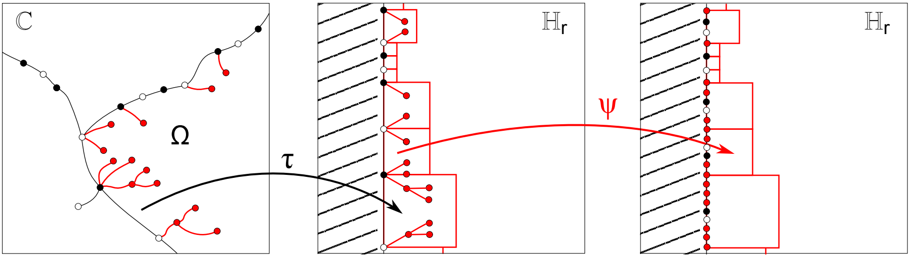

General case: $\forall$ edge $e \subset \partial\Omega \cup \partial\Omega'$, $\min\{\mathrm{diam}(\tau_{|\Omega}(e)), \mathrm{diam}(\tau_{|\Omega'}(e))\} \geqslant \pi$

We may assume $\tau_{|\Omega}(e) = i\Big[n\pi, n\pi + (2k+1)\pi\Big]$, $\forall$ edge $e \subset \partial\Omega$.

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$\square$

Generalization: Can we construct f with

- asymptotic values ?
- more critical values than only $\{-1, +1\}$?
- arbitrary high degree critical points ?

Solution: Let T be an infinite bipartite **graph**.

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The c.c. of $\mathbb{C}\setminus T$ are of three different types:

R-component: $\tau_{|\Omega} : \Omega \to \mathbb{H}_r$ conformally

L-component: $\tau_{|\Omega} : \Omega \to \mathbb{H}_\ell$ conformally

D-component: $\tau_{|\Omega} : \Omega \to \mathbb{D}$ conformally

Solution: Let T be an infinite bipartite **graph**.

More precisely:

R-component: $\Omega \xrightarrow{\tau_{|\Omega}} \mathbb{H}_r \xrightarrow{\cosh} \mathbb{C}\backslash[-1,+1]$

L-component: $(\Omega, \infty) \xrightarrow{\tau_{|\Omega}} (\mathbb{H}_\ell, -\infty) \xrightarrow{\exp} (\mathbb{D}, 0) \xrightarrow{\rho_\Omega} (\mathbb{D}, \rho_\Omega(0))$

D-component: $(\Omega, c_\Omega) \xrightarrow{\tau_{|\Omega}} (\mathbb{D}, 0) \xrightarrow{z \mapsto z^{d_\Omega}} (\mathbb{D}, 0) \xrightarrow{\rho_\Omega} (\mathbb{D}, \rho_\Omega(0))$

where $\rho_\Omega : \mathbb{D} \to \mathbb{D}$ is quasiconformal with $\rho_{\Omega|\partial\mathbb{D}} = \mathrm{Id}$.

Theorem 2 (Bishop 2011)

If (T, τ) satisfies the following conditions

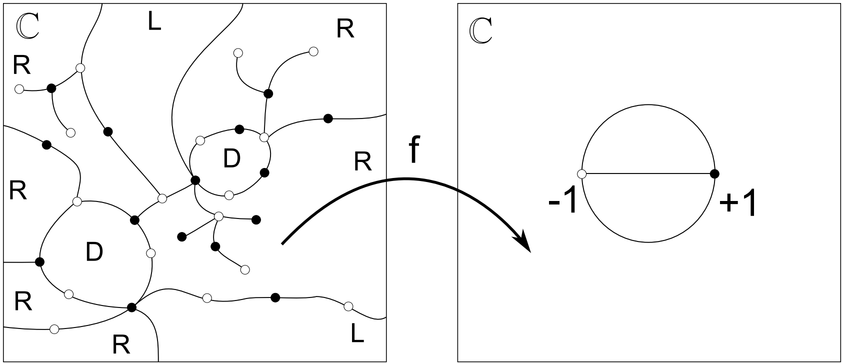
1. T has bounded geometry
2. L, D-components only share edges with R-components
3. on R-components, every edge has τ-size $\geqslant \pi$

then $\exists$ an entire function f in class $\mathcal{B}$ and a quasiconformal map ϕ /

$$f \circ \phi = \sigma \circ \tau \text{ off } T(r_0) \quad \text{with } \sigma(z) = \begin{cases} \cosh(z) & \text{on } R\text{-component} \\ \rho_\Omega(\exp(z)) & \text{on } L\text{-component} \\ \rho_\Omega(z^{d_\Omega}) & \text{on } D\text{-component} \end{cases}$$

Moreover

$$\begin{cases} \text{quasiconformal foldings only occur in } R\text{-components} \\ \text{the only asymptotic values of } f \text{ are in } \mathbb{D} \text{ (from } L\text{-components)} \\ \text{the only critical values of } f \text{ are in } \{-1, +1\} \cup \mathbb{D} \text{ (from } D\text{-components)} \\ f \text{ has critical points in } D\text{-components with arbitrary degree} \end{cases}$$



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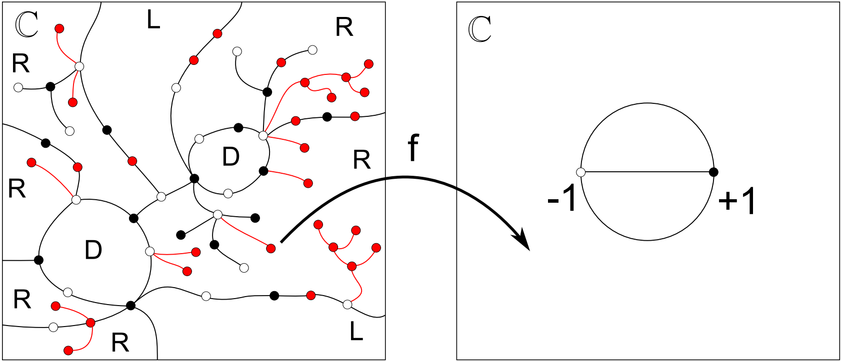

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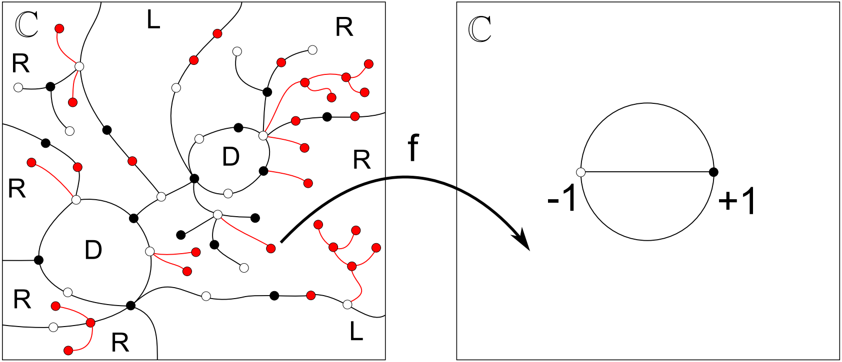

Moltes gràcies per la seva atenció.