Memorial Day in Honor of Jorge Sotomayor

slides for the talk by M. Zhitomirskii

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Constrained vector fields
(one of the problems of singularity theory)

$$A(x)x' = F(x), \quad x \in \mathbb{R}^n$$

Case of codim 0: $\det A(0) \neq 0, \; F(0) \neq 0$

Case of codim 1:

$\det A(x)$ vanishes at 0 in a regular way, and

$\ker A(0) \pitchfork S = \{x \in \mathbb{R}^n : \det A(x) = 0\}$, and

$F(0) \neq 0$ and $F(0) \notin \mathrm{Image}\, A(0)$

There are several cases of codim 2 and many cases of codim ≥ 3

Way: the system $A(x)x' = (\det A(x)) \cdot F(x)$ can be solved wrt x', we obtain a vector field E and should analyse the pair (E, S)

Two types of local classification problems

In a good part of local classification problems we can define:

$\kappa_1 =$ the number of functions of n variables one needs to parameterize the group of transformations defining the equivalence
(n if it is the group of local diffeos).

$\kappa_2 =$ the number of functions of n variables one needs to parameterize the space of germs under classification

$\kappa = \kappa_1 - \kappa_2 \geq 0 \Rightarrow$ singularity theory. Expectation:

a generic germ is finitely determined (its equivalence class is determined by the k-jet with a certain k); a generic germ has a group of symmetries parameterized by κ functions of n variables

$\kappa = \kappa_1 - \kappa_2 < 0 \Rightarrow$ local differential geometry. Expectation:

the case that a germ has non-trivial symmetries has codimension ∞; there exist normal forms parameterized by $|\kappa|$ functions of n variables and there are no normal form parameterized by a smaller number of functions of n variables

Example: (k, n)-distributions, $k \geq 2$ $= k$-dim subbundle of the tangent bundle $= k$-tuples of vector fields defined up to multiplication by a non-singular $k \times k$ matrix

$\kappa_1 = kn$, $\kappa_2 = k^2 + n$. $\kappa = kn - k^2 - n$ is ≥ 0 iff $k = n - 1$ or $(k, n) = (2, 4)$ which are the classical cases of Darboux and Engel.

If $k = n - 1$ then $\kappa = 1$, a generic germ is a contact distribution if n is odd, even-contact distribution if n is even. For odd n the symmetries of a generic germ are contactomorphisms, parameterized by one function of n variables; if n is even - even-contactomorphisms, parameterized by one function of n variables and one function of $n - 1$ variables.

$(k, n) = (2, 4)$ then $\kappa = 0$. Generic germ: Engel structure. Symmetries (Engelomorphisms) are parameterized by one function of 3 (not of 4 variables).

Therefore the local classification of (k, n) distributions with $k = n - 1$ and $(k, n) = (2, 4)$ are problems of singularity theory, and for other (k, n) with $k \geq 2$ - of local differential geometry

The first case, within distributions, when we have a local classification problem of local differential geometry, is (2,5) distributions:

a famous paper by E. Cartan of 1911, called "(2,5) variables paper", with Cartan method, to which in now-days are devoted international conferences in order to understand it better and use for other problems.

For (2,5) distributions $\kappa_1 = 2 \cdot 5 = 10, \kappa_2 = 4 + 5 = 9$ and one can expect that there is a normal form, for generic germs, parameterized by one function of 5 variables. People doing local differential geometry did not believe that such an exact normal form can be constructed. I did it using the Poincare method (in the same framework as the resonant normal form for vector fields) and quasi-homogeneous filtration wrt to the natural weights. Simultaneously I explained that the famous Cartan tensor (extremely difficult, not many people know what it is) is the first invariant in classification of quasi-jets, and complete invariant in classification of quaso-3-jets.

These results, appreciated by people in local differential geometry, allowed me to insist, in talks in conferences, that for problems in local differential geometry, where the 'deans of the faculty' are

S. Lie, E. Cartan, and N. Tanaka,

H. Poincare should be at least 'faculty member'.

Another example for 'promoting' H. Poincare is the most classical problem on local classification of Riemannian metrics = n-tuple of vector fields on $\mathbb{R}^n$ defined up to multiplication by a matrix in $SO(n)$. In this case $\kappa_1 = n^2, \ \kappa_2 = n(n-1)/2 + n$ and one can expect an exact normal form parameterized by $(n^2 - n)/2$ functions of n variables. Such normal form can be easily constructed by the Poincare method without any using of geodesics, normal coordinates, etc, with simulteneous explanation of the Rieman curvature tensor as the first invariant in the classification of usual jets. It is known (but not widely known) because this normal form holds in normal (geodesic) coordinates. But Poincare method is not only simpler, it gives much more, for example an exact normal form for conformal structures (Riemannian metrics defined up to multiplication by a non-vanishing function).

$(n^2 - n)/2$

Rules of behaviour

Singularity theory:

1. If case C adjoins case $\widehat{C}$ and you have not attacked case $\widehat{C}$ you have no right to attack case $\widehat{C}$

2. The bigger codimension of a case the less important it is. Case of ∞ codimension should not be attacked at all

3. For global objects on M^n consider their germs representing cases of codim $\leq n$ only (typical singularities) and cases of codim $n + d$ should be attacked in d-parameter families only

Local differential geometry:

The rules of behavior are principally different. The main interest is the cases when we have big symmetry groups - cases of infinite codimension. The other cases are studied by analyzing the obstakles for symmetries. Using normal forms is allowed but not popular.

Problems where definition of the codim of a case is a theorem

In some local classification problems ("between" singularity theory and local differentail geometry) the codimension of cases and adjaciences cannot be defined straightforwardly.

For example, when we classify germs of integrable (2,3) described by 1-form α such that $\alpha \wedge d\alpha \equiv 0$ and $\alpha(0) = 0$ (local foliation of M^3 with singularities) :

the case $d\alpha \equiv 0$ has codim 0 and not ∞, because it holds as soon as the complexification of α has isolated singularity (a part of Malgrange theorem)

and the case that α has non-isolated singularity also has codim 0 and not ∞, because it holds as soon as $d\alpha(0) \neq 0$ (Kupka theorem)

there is a very thin and involved bridge between Malgrange and Kupka cases (C. Camacho, A. Neto, other people) where the codim of a case cannot be defined, at least straightforwardly.

A vector field $x' = Ax + h.o.t.,\ x \in \mathbb{R}^n$
where there are infinitely many resonant relations between the eigenvalues of A might be linearizable.
Is this case, of codim ∞ important?
For whom? In which kind of works?

Namely this case occurs in

Theorem, R. Hartman, 1973. Any semi-simple representation in $gl(n)$ is 1-determined.

This theorem can be formulated as follows: assume that the span over $\mathbb{R}$ of germs at 0 of vector fields $V_1, ..., V_n,\ V_i(0) = 0$ is a semi-simple Lie algebra. Then $V_1, ..., V_n$ can be linearized simultaneously (even though in this case the linearization of each of the vector fields has infininitely many resonant relations).

Linearization of isotropy subalgebra of a transitive Lie algebra of vector fields

Local transitive Lie algebra = Lie algebra $\mathcal{A}$ of vector field germs at 0 such that $\mathcal{A}(0) = T_0\mathbb{R}^n$. If $\dim \mathcal{A} < \infty$, this means that $\mathcal{A}$ is spanned by $n + s$ vector field germs V_i such that $V_1(0) = \partial x_1, ..., V_n(0) = \partial x_n$.

The subslgebra $\mathcal{I} = \{V \in \mathcal{A},\ V(0) = 0\}$ is called isotropy subalgebra. The Lie algebra $j^1\mathcal{I}$ is called isotropy representation.

In local differential geometry a very important problem is distinguishing objects with non-trivial, preferably big transitive symmetry algebra (infinitesimal symmetries); such objects are colled homogeneous. Of course the isotropy subalgebra plays the key role.

Studying a number of examples, it was hard for me to find any case that the isotropy subalgebra $\mathcal{I}$ with faithful $j^1\mathcal{I}$ is not linearlizable. Faithful $j^1\mathcal{I}$ is an obvious necessary condition, it means that $\mathcal{I}$ does not contain vector fields with zero linear approximation. I found few examples for $\dim \mathcal{I} \geq 2$ but not for $\dim \mathcal{I} = 1$. In the analytic category, it is easy to prove that if $\dim \mathcal{I} = 1$ then $j^1\mathcal{I}$ is faithful.

Theorem. If a transitive Lie algebra of vector field germs on $\mathbb{R}^n$ has 1-dim isotropy subalgebra $\mathcal{I} = \mathrm{span}(V)$ then V is linearizable.

If $\dim \mathcal{I} \geq 2$, I have a necessary and sufficient (cohomological) necessary and sufficient condition under which $\mathcal{I}$ is linearizable. The corresponding theorem is based on the proposition that if one can "kill", by a change of coordinates, all terms of degree 1 then one can kill all terms of any degree.