Diagrams of Fold Mappings and Chains in $3D$ Filippov systems: a chaotic phenomenon coalition

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Preliminaries

Notations:

i) $A_0 = \{C^r \; mappings \; f : R^2, 0 \to R\}$
ii) $A = \{C^r \; mappings \; f : S^2 \to R\}$, r big enough and S^2 denotes the sphere of radius 1;
iii) $\chi_0 = \{germ \; of \; C^r v.f \; on \; S^2, \; f : S^2 \to R\}$
iv) $\Omega = \chi_0 \times \chi_0 - Filippov \; vector \; fields \; represented \; by \; Z = (X, Y)$
v) $I_0 = \{C^r \; involutions \; \alpha \; on \; (R^2, 0), \dim Fix\{\alpha\} = 1 \;\}$
vi) $B = \{C^r \; reversible \; mappings \; on \; S^2\}$

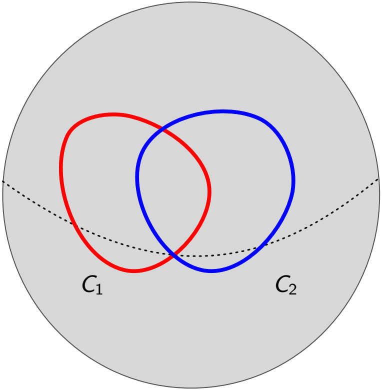
Recall that any $\varphi \in B$ is the composition of two involutions.

A fold mapping. Path of Fold Mappings. Normal form of a fold mapping.

Normal form of a $2D$ fold mapping: $f(x, y) = (x, y^2)$. The symmetric of f is $\varphi_f(x, y) = (x, -y)$ which satisfies $\varphi_f^2 = Id$ e $f \circ \varphi_f = f$.

Consider $f : S^2 \to R$ and $g : S^2 \to R$ s. t. $Sing\{f\} = C_1$, $Sing\{g\} = C_2$ are circles of fold singularities.

In S^2 - $C_1 \cap C_2 = \{p_1, p_2\}$, $p_1 \neq p_2$.



Remark

In the 80's Sotomayor exposed me to a research project on the classification of 2D applications through singular paths (singular sets). He later confirmed to me that this project did not progress. However, for me it was a source of inspiration.

Involution (dimension of Fixed set is 1) and Reversible Mappings - Local Approach.

Let $\varphi = \alpha \circ \beta$ be a reversible mapping in $R^2, 0$ (composition of two involutions) , with $\alpha(0) = \beta(0) = 0$.

0 is a hyperbolic fixed point (saddle type) only if the eigenvalues of $\varphi'(0)$ are real and of the form λ and $\frac{1}{\lambda}$.

Diagram of Fold Mappings.

$$D : \{R^2 \leftarrow_f R^2 \rightarrow_g \; R^2\}$$

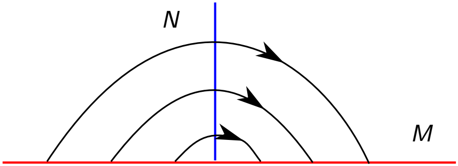
with associated involutions α_f and β_g and the associated reversible mapping

$$\phi = \alpha_f \circ \beta_g$$

A Construction.

Let X be a $3D$ C^r-vector field and M, N two $2D$ hyperplanes, with

$$0 \in M \cap N \textit{ and } M \pitchfork N \textit{ at } 0.$$



Assume that X is transverse to N at 0. We may, via Implicit Function Theorem, associate to X, a C^r-mapping,

$$f_X : M \to N$$

such that for each $p \in M$, $f_X(p)$ is the point where the trajectory γ_p of X through p reaches N.

When the contact of γ_0 with M is quadratic (fold singularity), then f_X is a fold mapping at $(M, 0)$.

In the general case, the following sets coincide:

i) $S_M(X)$ - tangency set between X and M.
ii) $Sing(f_X)$ singular set of f_X.

Remark

When p is a fold singularity of f_X then the Fixed Set of the associated involution (symmetric) coincides with $Sing(f_X)$.

Fold Singularity and fold-fold singularity

Let X be a $3D$ C^r- vector field defined in a compact region U.Fix, for simplicity, $h = x^2 + y^2 + z^2 - 1$ and $p \in R^3$ with $h(p) = 0$.

Definition

p is a *fold* singularity (or just fold) point of X provided that $Xh(p) = 0$ and $X^2 h(p) \neq 0$.

Remark

Vector Fields near the boundary of a 3-manifold , Lect. Notes in Math., 331, Springer Verlag, 1988.

We, Sotomayor and myself, spent a lot of energy on this article and it was of great value for my maturation in the Theory of Discontinuous Systems in Dimensions greater than 2.

But what I really want to emphasize is an article by Jorge, not very well known, little cited and which has an immeasurable scientific value:

Structural Stability in manifolds with boundary, in -Global Analysis and its Applications-, V. 3, pp. 167-176, Vienna (1974).

Fold-Fold singularity and T-singularity

Given a Filippov $Z = (X, Y)$, assume: $S^2 = \{h = 0\}$, p is a fold-fold singularity of Z

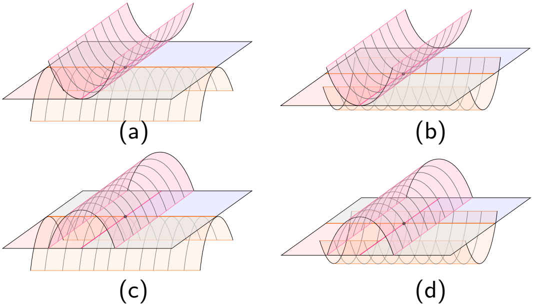


Figure: Fold-Fold Singularity: (a) Hyperbolic, (b,c) Parabolic and (d) Elliptic (T-Singularity).

Distinguished Regions of Σ: $XfYf \neq 0$

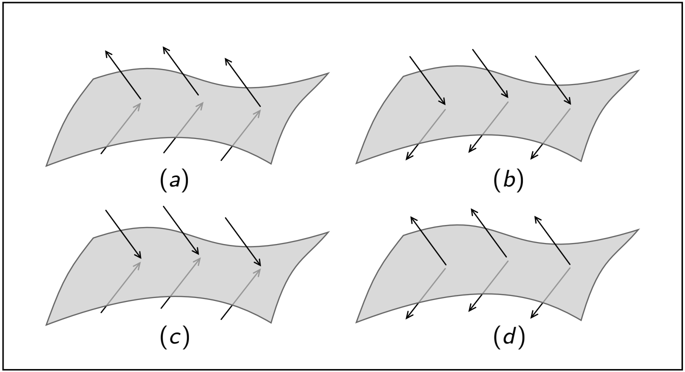


Figure: Regions in Σ: Σ^c in (a) and (b), Σ^{ss} in (c) and Σ^{us} in (d).

Filippov Solution

Denote $\Sigma^s = \Sigma^{ss} \cup \Sigma^{us}$.

On Σ^s, is defined the sliding vector field F_Z which is tangent to Σ.

T-Singularity

Fold-Fold Singularity

A point $p \in \Sigma$ is said to be a Σ-**singularity** of $Z = (X, Y)$ iff $Xf(p)Yf(p) = 0$. We say that p is a fold-fold singularity if

$$Xf(p) = Yf(p) = 0,\ X^2f(p)Y^2f(p) \neq 0,\ S_X \pitchfork S_Y \text{ at } p.$$

Elliptic Case - T-Singularity

If $p \in \Sigma$ satisfies that $Xf(p) = 0$, $Yf(p) = 0$, $X^2f(p) < 0$ and $Y^2f(p) > 0$, $S_X \pitchfork S_Y$ at p, then we say that p is a **T-singularity** or an **invisible fold-fold singularity** of $Z = (X, Y)$.

- It was an open problem since 1980 (works of M. A. Teixeira) and it was formalized by A. F. Filippov in 1988;
  A. F. Filippov. Differential equations with discontinuous righthand sides. Kluwer, 1988.

- M. Jeffrey et al. worked on this problem in a very nice intuitive way;
  A. Colombo and M. R. Jeffrey. The two-fold singularity of discontinuous vector fields. SIAM J. Applied Dynamical Systems, 8(2):624 640, 2009.

Ingredients to study the dynamics around a T-Singularity?

1. Involutions ϕ_X, ϕ_Y associated with X and Y, respectively;
2. Local behavior depends on F_Z and a (pseudo-) first return map $\phi = \phi_X \circ \phi_Y$;
3. Only one class of stability.

M. A. Teixeira and O. M. L. Gomide. Generic Singularities of 3D Piecewise Smooth Dynamical Systems, pages 373404. Springer International Publishing, Cham, 2018.

First Return Map

The composition of the fold mappings $\phi = \phi_X \circ \phi_Y$ works as a first return mapping associated to Z and Σ, with $\phi(0) = 0$.

The eigenvalues λ and μ of $d\phi(0)$ satisfy $\lambda.\mu = 1$.

Moreover, it satisfies either:

(C) λ and μ are complex and $|\lambda| = |\mu| = 1$, $\lambda, \mu = e^{\pm i\theta}$;

(S) $|\lambda| < 1 < |\mu|$ and 0 is a fixed point of type saddle of the diffeomorphism ϕ.

Case - (S) Auxilary mapping

The position of the local invariant manifolds of the saddle of ϕ can happen in the following way:

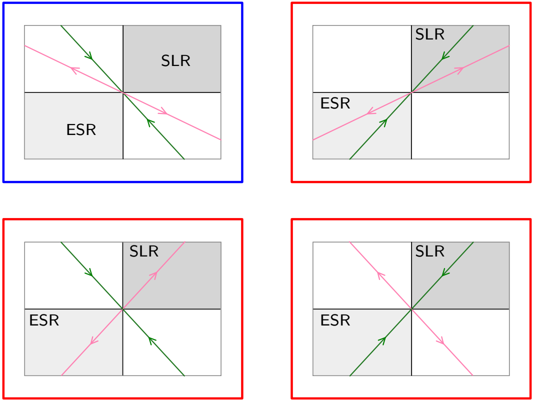
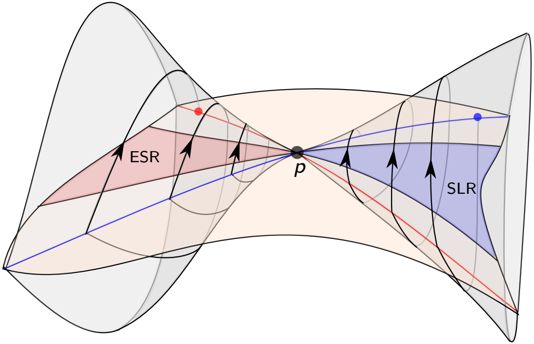


Figure: Position of W^s (pink) and W^u(green).

Case - (S) : Invariant Diabolo

If $W^s, W^u \subset \mathrm{SWR}$, then we have a nonsmooth invariant diabolo:



Notice that it isolates the dynamics in SLR and ESR.

Theorem

Z_0 is locally SS at a T-singularity $p \Leftrightarrow$ its first return map ϕ_0 has a saddle at p with both local invariant manifolds in Σ^c.

Global Connections

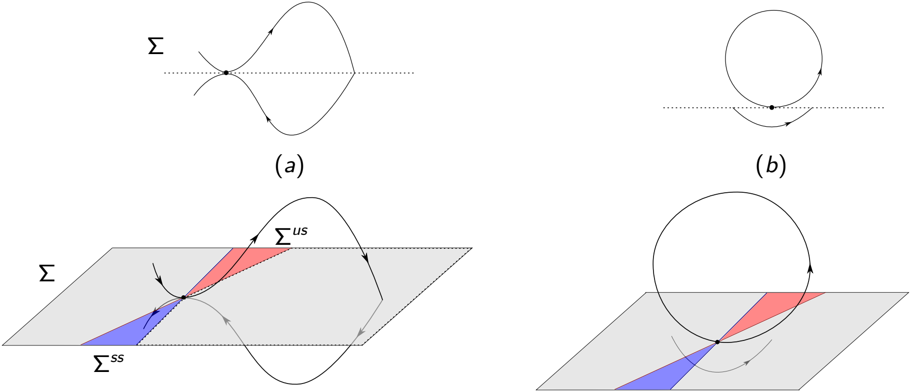
What kind of dynamics is originated from the global extension of these local invariant manifolds?

Definition

Let $Z_0 = (X_0, Y_0) \in \Omega$ having fold-fold singularities $p_0, q_0 \in \Sigma$ ($p_0 = q_0$ is also considered) and let $-\infty \leq a < b \leq \infty$.
An oriented piecewise smooth curve $\Gamma : (a, b) \to \mathbb{R}^3$ is said to be a **fold-fold connection** of Z_0 between p_0 and q_0 if it satisfies the following conditions.

- i) $\mathrm{Im}(\Gamma) \cap \Sigma^+$ (resp. $\mathrm{Im}(\Gamma) \cap \Sigma^-$) is a union of orbits of X_0 (resp. Y_0).
- ii) $\mathrm{Im}(\Gamma) \cap \Sigma \subset \Sigma^c$.
- iii) $\lim_{t \to a} \Gamma(t) = p_0$ and $\lim_{t \to b} \Gamma(t) = q_0$.

$p_0 = q_0 =$ T-singularity + Z_0 locally SS at p_0= **T-Chain**



(a) (b) (c) (d)

T-Chains

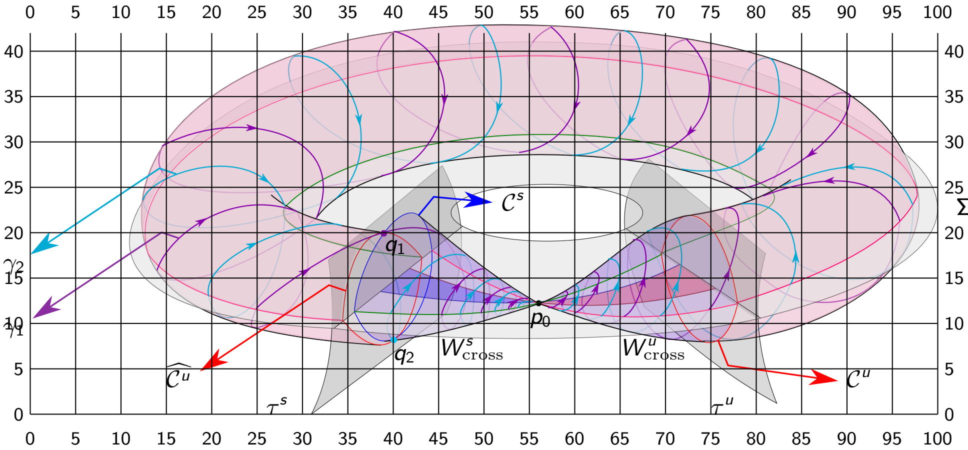


Figure: A Filippov system Z_0 satisfying (TC) and (R) conditions having two T-chains γ_1 and γ_2 passing through q_1 and q_2, respectively.

Example

$$Z_\epsilon = \begin{cases} \dfrac{X_2+X_1}{2} + \phi\left(\dfrac{g}{\epsilon}\right)\dfrac{X_2-X_1}{2} & if \quad z>0 \\ \dfrac{Y_2+Y_1}{2} + \phi\left(\dfrac{g}{\epsilon}\right)\dfrac{Y_2-Y_1}{2} & if \quad z<0, \end{cases}$$

where

$$\phi(x) = \begin{cases} -1, & \text{if } x < -1, \\ \sin(\pi x/2), & \text{if } |x| \leq 1, \\ 1, & \text{if } x > 1. \end{cases} \tag{1}$$

$g(x,y,z) = -(y + 17/10x - 5/2)$, $X_1(x,y,z) = (1,-1,y)$, $Y_1(x,y,z) = (-1,2,-x)$, $X_2(x,y,z) = (1,-1,y-2)$ and $Y_2(x,y,z) = (-1,3,-(x-2))$.

Main Results

Theorem

Let $Z_0 \in \Omega$, having a two two distinct T-chains of Z_0 at p_0 and satisfying some generic conditions at p_0 in Σ. Then:

1. *There exists $n \in \mathcal{N}$ such that Φ_0^n admits a Smale horseshoe Δ in U.*
2. *The hyperbolic invariant set Λ in the horseshoe Δ always contains a point $\widehat{q}_i \in \gamma_i \cap \Sigma$, for $i = 1, 2$.*
3. *Every orbit of Z_0 passing though a point of Λ is a crossing orbit.*

Main Results

As a direct consequence:

Theorem

Moreover, the following statements hold.

1. *There exists an infinity of closed crossing orbits* Γ *of* Z_0*, which are of saddle type (i.e. the first return map of* Z_0 *associated to* Γ *has a hyperbolic fixed point of saddle type);*
2. *There exists an infinity of non-closed crossing orbits* Γ *of* Z_0*;*
3. *There exists a crossing orbit* Γ_d *of* Z_0 *such that* $\Gamma_d \cap \Lambda$ *is dense in* Λ*.*

A Bridge between Chaos and Divergent Diagrams

Consider a divergent diagram $\{f, g\}$ of C^r fold mappings:

$$R^2 \leftarrow_f : U : \rightarrow_g R^2$$

with U being an open set in R^2.

We know that if two divergent diagrams of folds $\{f_0, g_0\}$ and $\{f, g\}$ are equivalent by homeomorphisms h in the source and k in the target then h is a simultaneous equivalence between the pairs $(\varphi_{f_0}, \varphi_{g_0})$ and (φ_f, φ_g). Moreover h realizes an equivalence between the compositions $\varphi_{f_0} \circ \varphi_{g0}$ and $\varphi_f \circ \varphi_g$.

A Bridge between Chaos and Divergent Diagrams

Given a divergent diagrams of folds (f, g) let D_f and D_g be the global singular sets of f and g, respectivey (in general position) st:

i) $\phi = \alpha_f \circ \alpha_g$ is well defined in a compact region;
ii) $D_f \cap D_g$ are exacty two Structurally Stable T-singularities p and q
iii) $W^s(p)$ and $W^u(q)$ intercept transversally (Heteroclinic Reversible Connection) (Amadeo) (iv- When $p = q$, a homoclinic reversible connection)

Under the above hypotheses, a Horseshoe is created for $f \circ g$.

References

K. S. Andrade, O. M. L. Gomide, and D. D. Novaes. Qualitative analysis of polycycles of filippov systems. Preprint ArXiv: 1905.11950.

M. Guardia, T. Seara, and M. Teixeira. Generic bifurcations of low codimension of planar filippov systems. Journal of Differential Equations, 250(4):1967 2023, 2011.

M. A. Teixeira and O. M. L. Gomide. Chains in 3D Filippov systems: A chaotic phenomenon. Journal de Mathmatiques Pures et Appliques 159, 168-195

S. Wiggins. Global Bifurcations and Chaos: Analytical Methods. Springer-Verlag, 1988.

His work is already immortalized and now we can only learn from his memories and perpetuate our admiration. If I lost a great longtime friend, with whom I had the pleasure of sharing several unforgettable moments, and for Brazilian and Peruvian mathematics, it represents the loss of one of its most prestigious representatives.