Sotomayor and the bifurcations

Memorial Day

March 25, 2022

[S1] J. Sotomayor *Structural stability of first order and Banach varieties,* (Portuguese) Univ. Nac. Ingen. Inst. Mat. Puras Apl. Notas Mat. 4 (1966), 11-52.

[S2] J. Sotomayor *Generic one-parameter families of vector fields on two-dimensional manifolds,* Bull. Amer. Math. Soc. 74 (1968), 722-726

[S3] J. Sotomayor *Structural stability and bifurcation theory.* Dynamical systems, (Proc. Sympos., Univ. Bahia, Salvador, 1971), Academic Press, New York, (1973), 549-560.

[S4] J. Sotomayor *Generic one-parameter families of vector fields on two-dimensional manifolds,* Inst. Hautes Etudes Sci. Publ. Math. No. 43 (1974), 5-46.

[DRS1] F. Dumortier, R. Roussarie, J. Sotomayor *Generic 3-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case of codimension 3*, Erg. Th. and Dyn. Sys., vol. 7 (1987) 375-413.

[DRS2] F. Dumortier, R. Roussarie, J. Sotomayor *Generic 3-parameter families of vector fields unfoldings of saddle, focus and elliptic singularities with nilpotent linear part*, in : Lect. Notes in Math., n$^\circ 1480$, "Bifurcations of planar vector fields : nilpotent singularities and abelian integrals" (1991) 1-164.

[DRS3] F. Dumortier, R. Roussarie, J. Sotomayor *Bifurcations of Cuspidal loops*, Nonlinearity Vol. 10 (1997) 1369-1408

One-parameter families

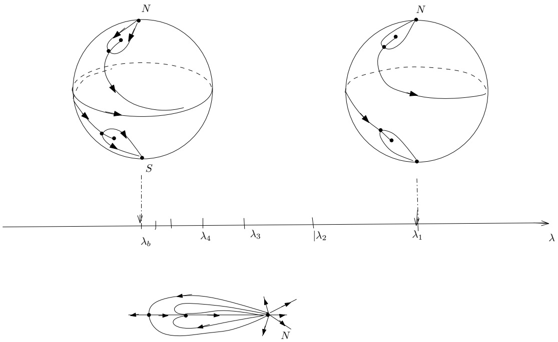
In the 50', **Peixoto** : **the set Σ_0 of stucturally stable vector fields on compact surfaces.**
In the 60', **Sotomayor** : **generic one-parameter families of vector fields,** i.e. the simplest way to break the structural stability.
Based on the general frame introduced by **Thom** and developped by **Mather** : **genericity, transversality and so on.**

Principal idea : to obtain, if possible, a stratification of the functional space, where the stratum Σ_k**, a submanifold of codimension k,** is the set of objects of codimension k, appearing **in generic families with k parameters.**

Sotomayor made the first step of this program for the space $\mathcal{X}$ of **vector fields on a compact surface,** finding the stratum **Σ_1 of codimension 1.**
This stratum is made by the vector fields where the conditions of Peixoto are just violated one time : the vector field has a unique **semi-hyperbolic singular point** or **saddle connection,** or **Hopf singular point** or **semi-stable limit cycle** and so on.
These vector fields are **stable at first order** (i.e. in restriction to Σ_1).

Principal result (see [S4]) :
(1) Σ_1 is an **immersed submanifold** of codimension 1 in $\mathcal{X}$.
(2) One-parameter families **tranverse to Σ_1** are "structurally stable" in the sense of **fiber-$\mathcal{C}^0$ equivalence.**

The fact that Σ_1 is an immersed submanifold, but **not imbedded,** is illustrated by the following generic family X_λ :



The famity cuts Σ_1 at the values $0, \lambda_1, \lambda_2, \ldots$. The sequence $(\lambda_n), n \geq 1$, tends to 0. X_0 has a semi-stable closed orbit and X_{λ_n} has a saddle connection spiraling n times along the equator. This means that Σ_1 contains a sequence of submanifolds (Σ_1^n) coverging toward another submanifold Σ_1^0 also contained into Σ_1

Unfolding the singularities of codimension 3

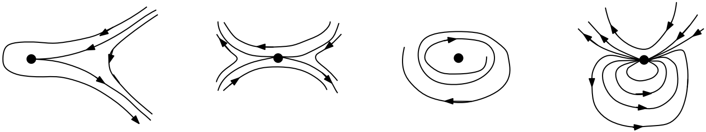
In the $80'$ **Dumortier and myself :** the project to complete the study the **generic unfoldings of codimension-3 singularities of planar vector fields**.

When we spoke of this project to **Sotomayor,** he seemed very interested. In fact we understood that **he had already looked at the question and had imagined some possible diagrams of bifurcation** (diagrams, already proposed without proofs, by **Bazikin, Kuznietzov** and **Khibnik** in 1985). This explained our collaboration on the two works [DRS1] and [DRS2].

At this time, many 3-cod. singularities were already well studied such as semi-hyperbolic points, Hopf-Takens singularities. It remained four cases to study :

Cusp, saddle, elliptic and focus singularities of cod. 3

The singularities



Cusp case Saddle case Focus case ($\varepsilon_2 = 1$) Elliptic case

These four generic singularities are the germs with 4-jets equivalent to :

(1) Cusp case : $y\frac{\partial}{\partial x} + (x^2 \pm x^3 y)\frac{\partial}{\partial y}$ This defines the submanifolds $\Sigma^3_{C\pm}$ of codimension 3 in the space of germs.

For the three other cases :

$$y\frac{\partial}{\partial x} + (\varepsilon_1 x^3 + bxy + \varepsilon_2 x^2 y + f x^3 y)\frac{\partial}{\partial y}$$

with $b > 0, \varepsilon_{1,2} = \pm$ and any f.

(2) Saddle case : $\varepsilon_1 = 1$, any ε_2 and b. This defines the submanifolds $\Sigma^3_{S\pm}$

(3) Focus case :$\varepsilon_1 = -1$, any ε_2 and $0 < b < 2\sqrt{2}$. This defines the submanifolds $\Sigma^3_{F\pm}$

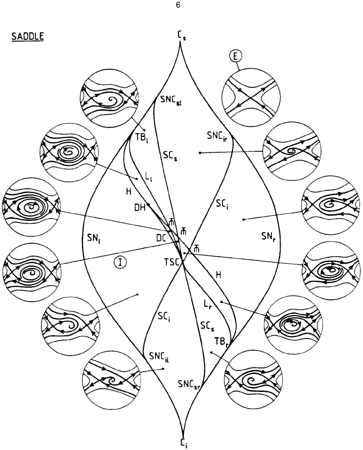
(4) Elliptic case : $\varepsilon_1 = -1$, any ε_2 and $b > 2\sqrt{2}$. This defines the submanifolds $\Sigma^3_{E\pm}$

The result

One defines **standard 3-parameter unfoldings** by
$y\frac{\partial}{\partial x} + \left(x^2 + \mu + y(\nu_0 + \nu_1 x \pm x^3)\right)\frac{\partial}{\partial y}$ in the cusp case and by
$y\frac{\partial}{\partial x} + \left(\varepsilon_1 x^3 + \mu_2 x + \mu_1 + y(\nu + bx + \varepsilon_2 x^2 +)\right)\frac{\partial}{\partial y}$ in the three other cases.

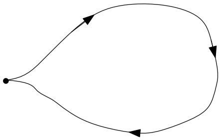
Partially proved claim in [DRS2] :
Let X_λ and Y_λ be two local 3-parameter families with X_0, Y_0 belonging to the same set $\Sigma^3_{C\pm}$, $\Sigma^3_{S\pm}$, $\Sigma^3_{F\pm}$ or $\Sigma^3_{E\pm}$. Suppose that both families are transversal to this set. Then, they are fiber-$\mathcal{C}^0$ equivalent.
The result is completely proved for $\Sigma^3_{C\pm}$ but depends on open conjectures about generalized Abelian integrals in the three other cases.

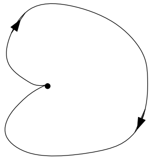
Bifurcation diagram for the saddle singularity



Generic unfolding of the cuspidal loop

It is the subject of [DRS3]. Again a 3-parameter unfolding, but now of a global object, and then more in the spirit of the first works of **Sotomayor. One connects the two separatrices of a cusp singularity** (studied previously by **Bogdanov** and **Takens**). The cuspidal loop is clearly an object of codimension 3.



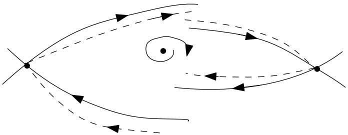


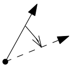
A complete result for generic 3-parameter unfoldings depends an stll open conjecture about integral functions.

Steps of study and tools

(1) Singular points. Bifurcating singular points are given by an **algebraic algorithm.** Easily solved in our case.

(2) Rotating vector fields. The vector field **rotates in the same direction at each point** in function of some parameter :





(3) Normal form for singularity unfoldings : a simplifyed writting of a finite jet, interpreted as a **principal part** which depends on **versal parameters** (parameters essential for the next blown up step). This is obtained using some **generic conditions.**

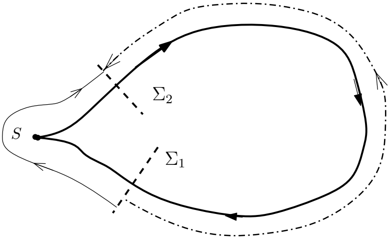
(4) Two types of blowing-up :

(a) Rescaling : Introduced by **Bogdanov** and **Takens** for the cusp and also used by ourselves for the four singularities.

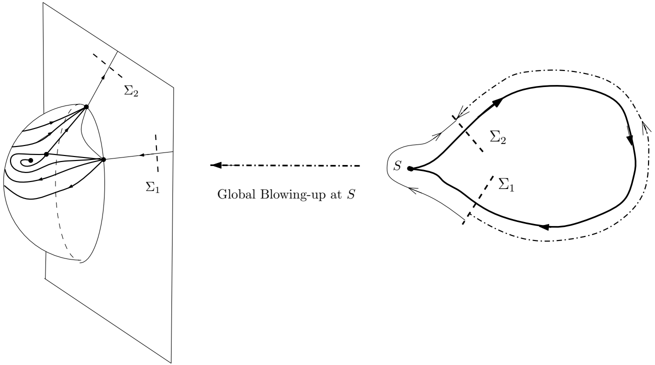
A difficulty : As the rescaling is singular at the origin, the **domain of study shrinks to a point** when a radial parameter goes to 0.

One has to extend the study in a fixed domain by using for instance the Poincaré-Bendixson Theorem.

The shrinking effect in the rescaling prevents from **gluing the study near a singular point to an external study.** For instance in cuspidal loop case :



(b) Global blowing-up : A **blowing-up at the origin of the product of the phase-space by the parameter space.**



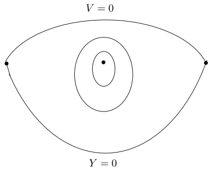
The domain of the global blowing-up contains the rescaling domain as a chart and also other charts which **connect the rescaling domain with the exterior**. The price to pay : the family is replaced by a **singular foliation.**

(5) (Generalized) Abelian Integrals are used to detect the limit cycles bifurcating from Hamiltonian closed levels. Such a **perturbation of Hamiltonian** may appear after blowing up.

It is the case in [DRS1] with a **true Abelian integral**. This Abelian integral has a **Chebyshev property** as it was proved by **Petrov.** This property implies a complete result for the bifurcation problem.

For the three other cases, in [DRS2], one has **generalized Abelian integrals.**

For instance, in the **saddle case** one has :
An Integrating factor : $K = V^{\alpha-1}Y^{\beta-1}$, an Hamiltonian
$H = -\frac{1}{\alpha+\beta}V^{\alpha}Y^{\beta}$
where $\alpha = \frac{1}{4}(b+\sqrt{b^2+8}), \beta = \frac{1}{4}(b-\sqrt{b^2+8})$, and
$V = y - \alpha(x^2-1), Y = \beta(x^2-1) - y$
There is a **center region filled by Hamiltonian cycles**
$\gamma_h = \{\mathbb{H} = h\}$, with $h_0 < h < 0$ for $h_0 = H(0;0) < 0$.



The integral function :

$$\int_{\gamma_h} K(\mu + \nu y + x^2 y)dx = \mu J_0(h) + \nu J_1(h) + J_2(h)$$

The claim is that **the curve $h \to (\frac{J_1}{J_0}, \frac{J_2}{J_0})$ is strictly convex.** The second part of [DRS2], written by **H. Zołandek,** is devoted to a proof of this claim and similar results for the other cases. Unfortunately, it appears later a mistake in the proof. Then, **the question remains open,** the reason why the results for bifurcation remains just partially proved.

It is even worst in [DRS3] where the integral function to study is the **finite time** to go from infinity to infinity, along **non closed orbits.** A *"numerical evidence"* for the properties of this time-function was obtained by or **C. Simó.**