Classical differential geometry: a singular point of view

Memorial Day in honor of Jorge Sotomayor

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Soto was not content with making discoveries, he also made students, which is sometimes better…

“If we wish to foresee the future of mathematics,

our proper course is to study the history and

present condition of the science.”

H. Poincaré, The Future of Mathematics, lecture read by G. Darboux, Rome, ICM, 1908

Historical landmarks

1707  1783 *Founder of differential geometry on surface (1767)*

Euler

Normal curvature,
principal curvature,
principal directions.

1746  1818 *Founder of the global study of lines of curvature (1796)*

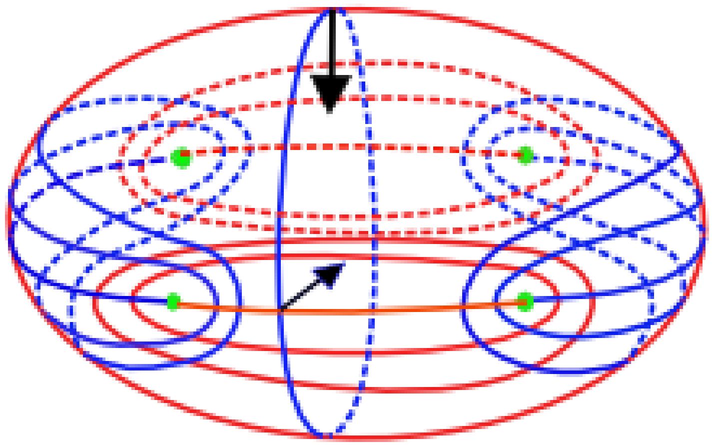
Monge

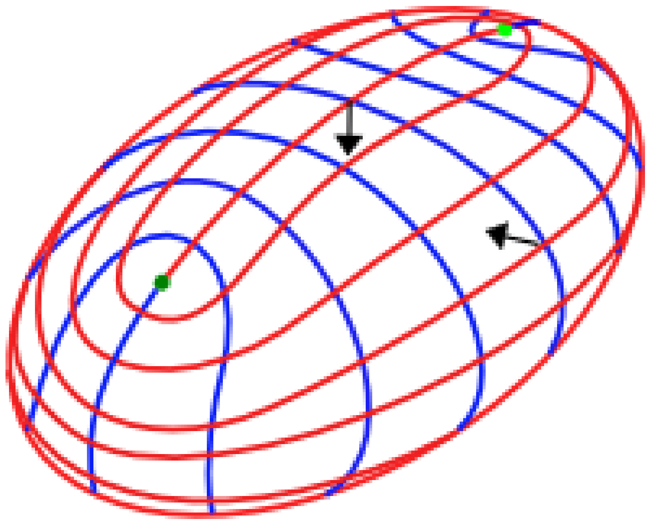
Analyst and geometer

G. Monge (1796) - Linhas de Curvatura no Elipsóide

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Coordenadas dos pontos umbílicos: $$\left(\pm a\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, 0, \pm c\sqrt{\frac{c^2 - b^2}{c^2 - a^2}}\right)$$





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Monge

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First global theory for the lines of curvature

1784  1873

Dupin

Triply orthogonal system (1813)

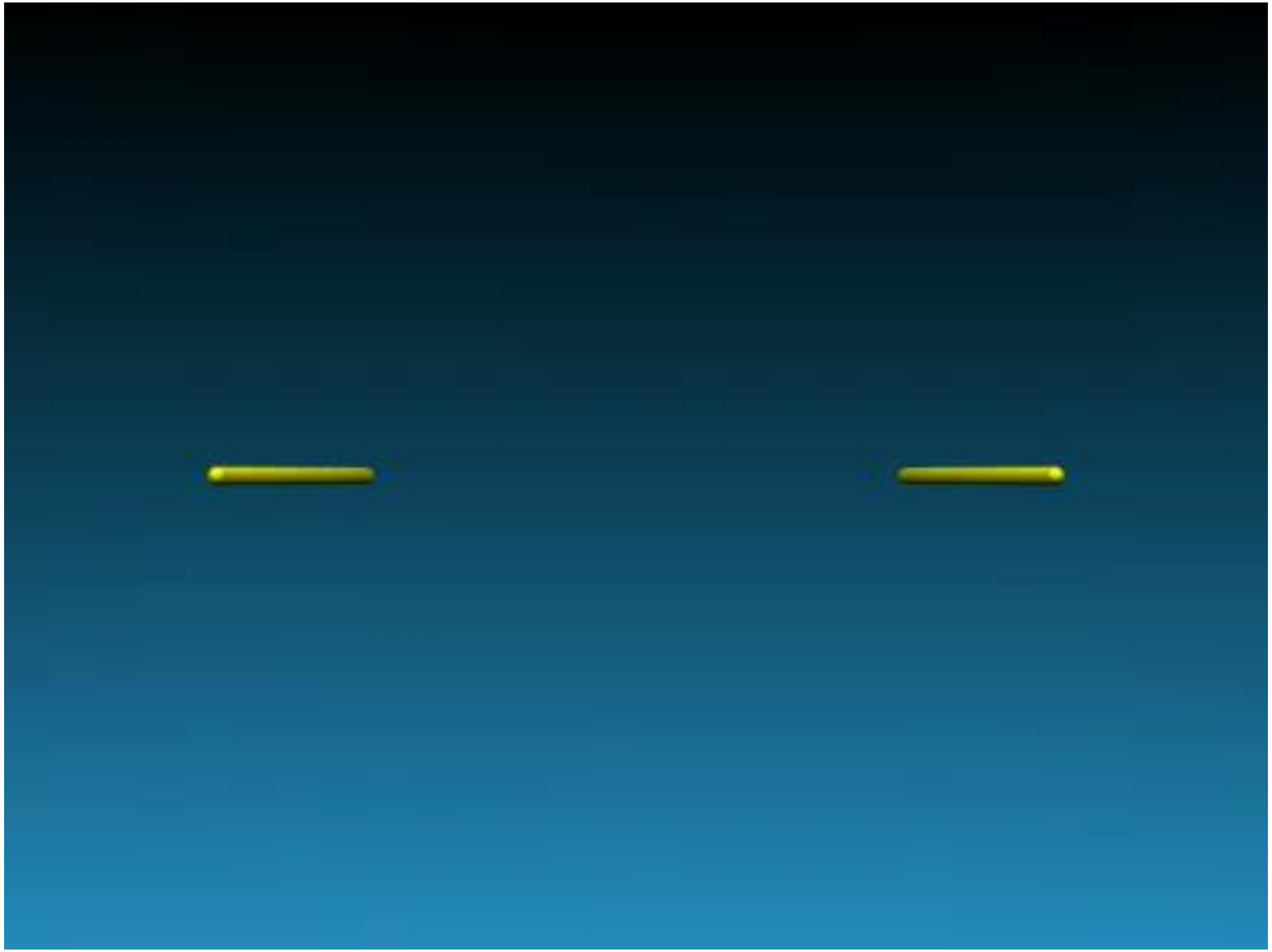
1784

First global theory for the lines of curvature

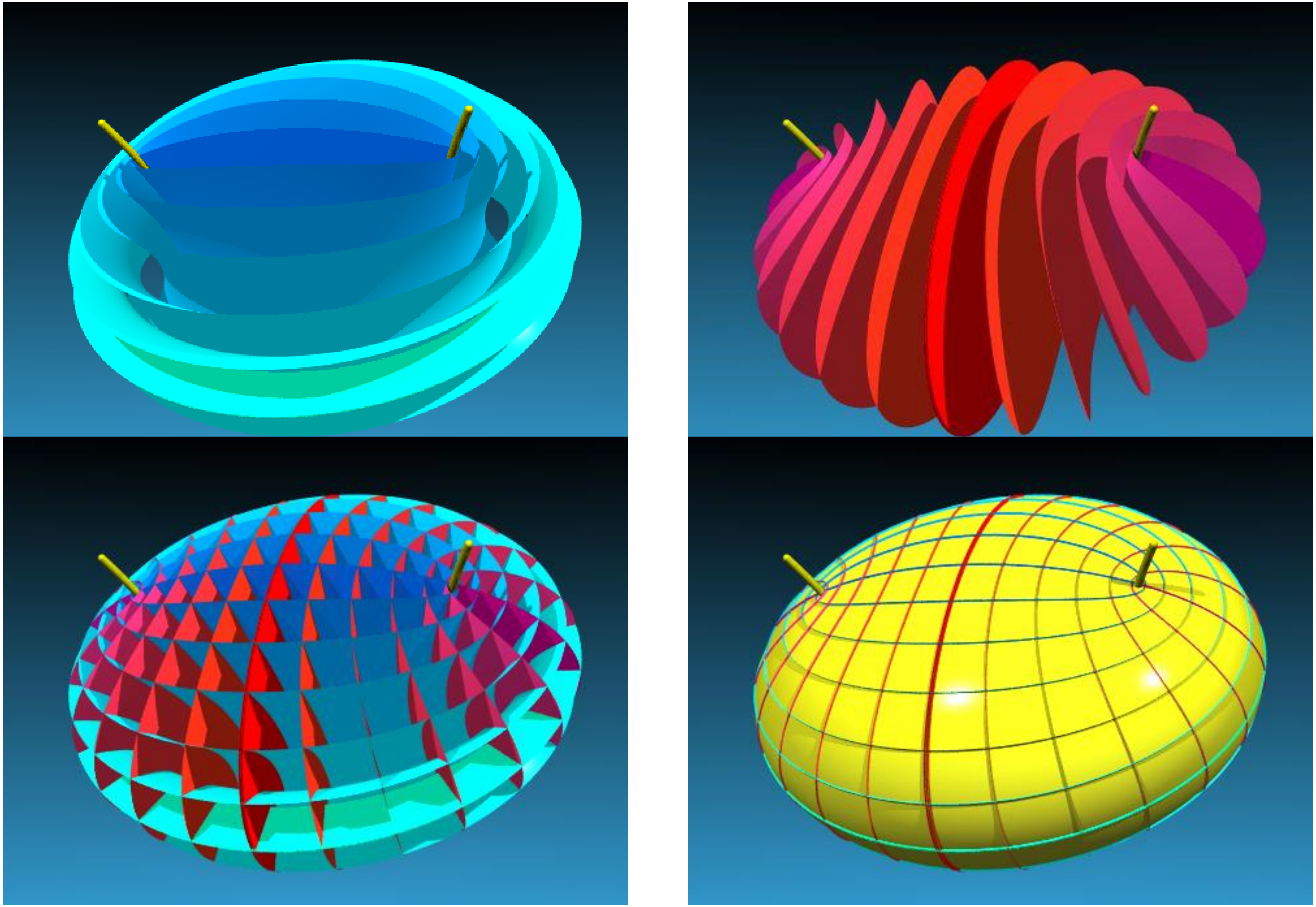
1873

Dupin

Dupin's Theorem: *Triply orthogonal families of Surfaces intersect along lines of curvature.*



https://images.math.cnrs.fr/Un-systeme-triple-orthogonal.html?lang=fr



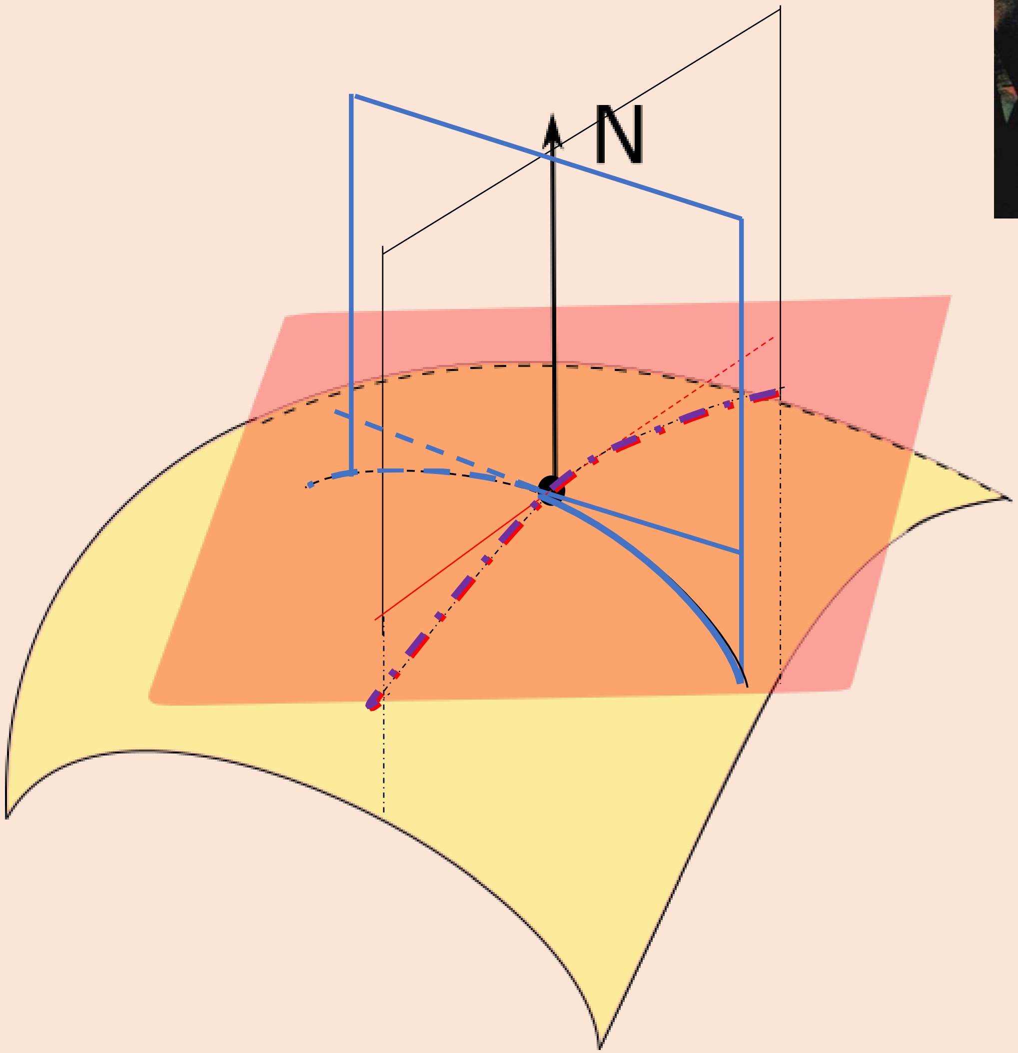
1707

Founder of differential geometry on surface (1767)

1783

Euler

Fundador da geometria

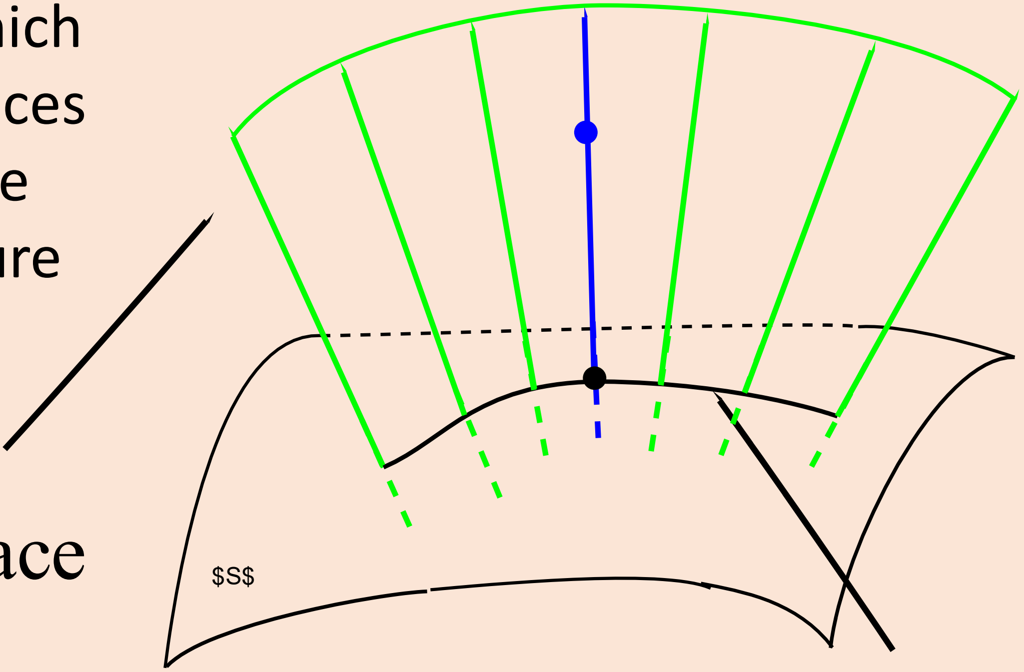


1746 **Fundador do estudo global da linhas de curvatura (1796)** 1818

Monge

Análise, Geometria

The normal sections for which this ruled surfaces are developable are the curvature lines.



Ruled surface

Y=x+tN

Normal section

28º Colóquio de la Sociedad Matmética Peuana, Agosto 2010 (J. Sotomayor and R. Garcia)

How was Monge taken to the study of principal lines?

What about the study of ruled and developable surfaces and line congruence?

Linhas Principais em Superfícies

Como G. Monge foi conduzido ao estudo das linhas principais?

E ao estudo de superfícies regradas e desenvolvíveis e congruências de retas?

PROBLEMA DO TRANSPORTE: Transportar uma quantidade fixa de matéria preservando a massa e minimizando o momento de massa (massa vezes deslocamento).

G. Monge, Mémoire sur la théorie des déblais et des remblais, Mémoires de l'Acad. des Sciences, 666-704, (1781).

Ronaldo Garcia (IME/UFG) e Jorge Sotomayor (IME/USP

Teoria Qualitativa das E.D.O. da Geometria Clássica

R. Garcia, D. Lopes and J. Sotomayor, ICM 2018

Line Congruence

A line congruence is defined by a pair

$$(x, \xi) \in C^{\infty}(U, R^3 \times R^3 \backslash \{0\}\,\})$$

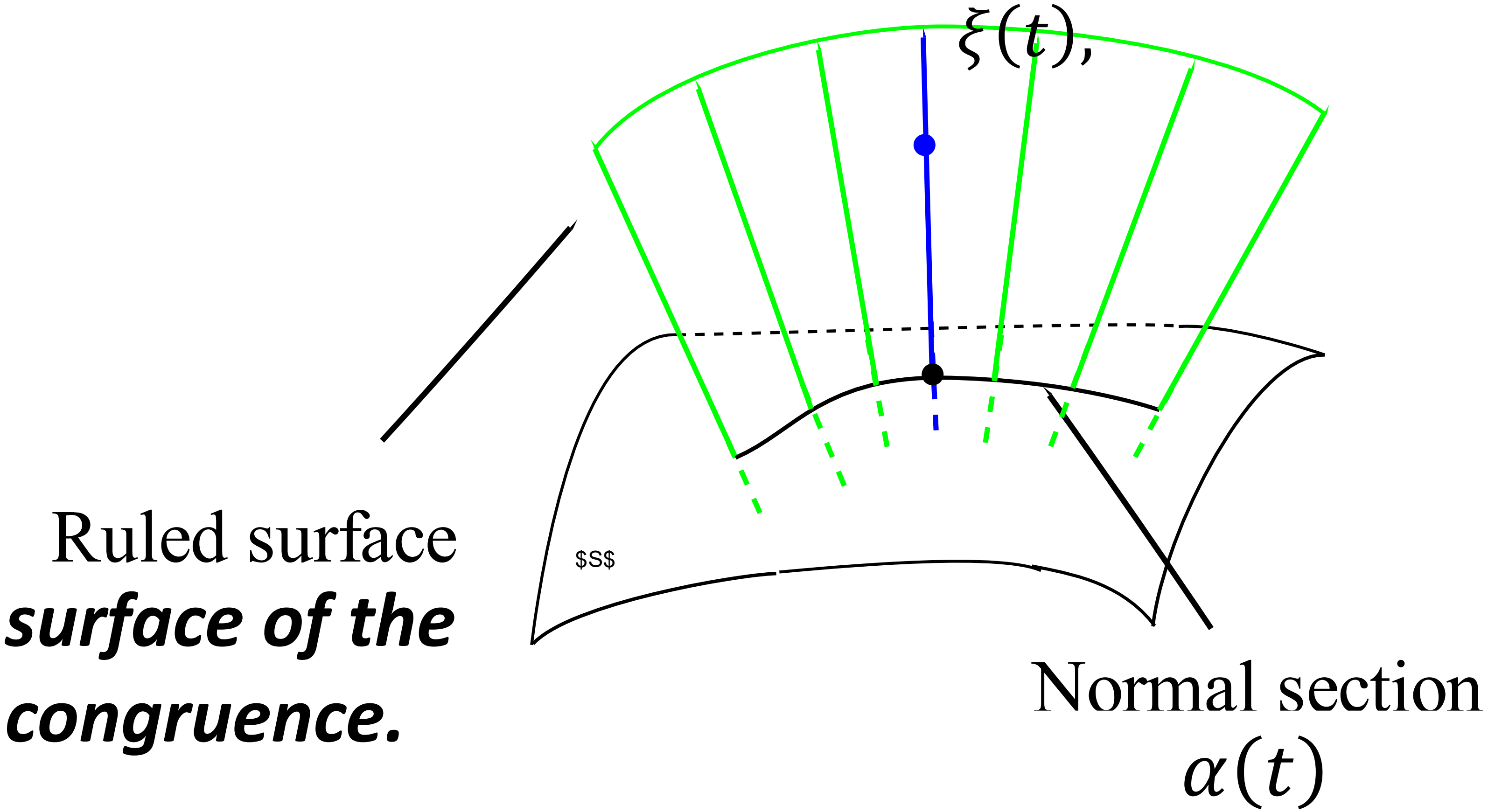
where $U \subset R^2$ is open and for all $(u_1, u_2) \in U$, $\xi(u_1, u_2)$ denotes the direction of the line through the point x (u_1,u_{2}) .

An example of line congruences is given by the normal lines of a surface of R^3.

Definição: The lines of the congruence which pass through a curve $\alpha(t)$ on the reference surface $x(u,v)$ form a ruled surface

$$Y(t,w) = \alpha(t) + w\,\xi(t), t \in I, w \in \mathrm{R},$$

called ***surface of the congruence.***



When x is normal to ξ, the congruence $\{x, \xi\}$ is called exact normal line congruence.

Proposition: Let $\{x, \xi\}$ be an exact normal line congruence. A curve C on the reference surface parametrized by $\alpha\colon\ I\ \to R^3$is a line of curvature if and only if the surface of the congruence $Y(t, w) = \alpha(t) + w\xi(t)$ is developable.

What about the study of ruled and developable surfaces and line congruence on singular surface?

Is it possible define line curvature on singular surface using Monge approach?
Or Dupin approach?

Tohoku Math. J.
52 (2000), 163–172

LINES OF PRINCIPAL CURVATURE AROUND UMBILICS AND WHITNEY UMBRELLAS

RONALDO GARCIA, CARLOS GUTIERREZ AND JORGE SOTOMAYOR

(Received February 26, 1998, revised November 4, 1999)

Abstract. In this paper is studied the configuration of lines of curvature near a Whitney umbrella which is the unique stable singularity for maps of surfaces into $\boldsymbol{R}^3$. The pattern of such configuration is established and characterized in terms of the 3-jet of the map. The result is used to establish an expression for the Euler-Poincaré characteristic in terms of the number of umbilics and umbrellas.

1. Introduction. The bending or curvature pattern of a smooth mapping $\alpha : M \to \boldsymbol{R}^3$, where M is a compact oriented two dimensional manifold, will be represented here by singular points, $\mathcal{S}_\alpha$, at which the mapping has rank less than 2 and the bending can be regarded to be infinite; the umbilic points, $\mathcal{U}_\alpha$, at which the bending is finite but equal in all directions: and by the family of lines of principal curvature $\mathcal{F}_{1,\alpha}$ and $\mathcal{F}_{2,\alpha}$ defined on $M \setminus (\mathcal{U}_\alpha \cup \mathcal{S}_\alpha)$, which represent the directions along which the bending, quantitatively expressed by the *normal curvature*, is extremal (maximal along $\mathcal{F}_{1,\alpha}$ and minimal along $\mathcal{F}_{2,\alpha}$). These four objects will be assembled into the *principal configuration* of the mapping, denoted by $\mathcal{P}_\alpha = (\mathcal{S}_\alpha, \mathcal{U}_\alpha, \mathcal{F}_{1,\alpha}, \mathcal{F}_{2,\alpha})$. The points of $\mathcal{S}_\alpha$ and $\mathcal{U}_\alpha$ are regarded as the singularities of the foliations $\mathcal{F}_{1,\alpha}$ and $\mathcal{F}_{2,\alpha}$.

The study of these foliations near umbilic singularities was started by Darboux [Dar], in the class of analytic surfaces. Under generic conditions on the third derivatives, he found three types, D_1, D_2 and D_3, called here *Darbouxian Umbilics*. These points are illustrated in Figure 3.

This result of Darboux was rediscovered and reproved by Gutierrez and Sotomayor, [GS1]–[GS3], in the context of structural stability of principal lines on regularly immersed surfaces of class C^r, $r \geq 4$. They showed that Darbouxian umbilic points characterize those with local structurally stable configuration, under small C^3 deformations of the surface. See

LINE CONGRUENCE IN SINGULAR SURFACES IN $\mathbb{R}^3$

D. LOPES, I. C. SANTOS, M. A. S. RUAS, T. MEDINA

ABSTRACT. This paper establishes the geometric structure of the line congruences in a singular surface in $\mathbb{R}^3$...

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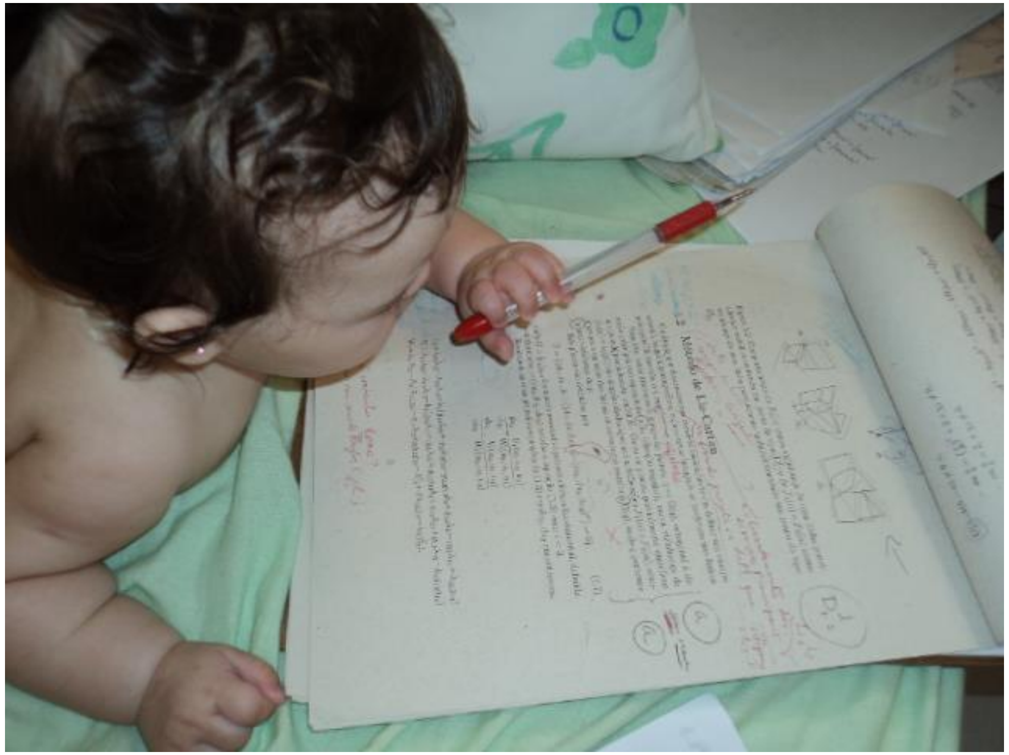
(USP/São Carlos)

T. Medina
(Postdoc USP/São Carlos)

To conclude, soto wasn’t only my PhD adviser, he taught me to see the beauty of mathematics and to look at a math problem for what it is... without a specific area.

Soto was also an amazing human being: Marilda must remember when I had my first daughter during my PhD and Soto and Marilda came to visit me, and we talked about the thesis in a super productive way... making me even more excited to finish the my PhD...



Thank you soto, our “Mestre MAYOR”

IMAGES DES MATHÉMATIQUES

La recherche mathématique en mots et en images

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GASPARD MONGE, le beau, l'utile et le vrai
Le *24 décembre 2011* - Ecrit par [Étienne Ghys]
https://images.math.cnrs.fr/Gaspard-Monge.html?lang=fr#nb36

UN SYSTÈME TRIPLE ORTHOGONAL, une figure classique de géométrie différentielle
Le *25 novembre 2008* - Ecrit par [Étienne Ghys], [Jos Leys]
https://images.math.cnrs.fr/Un-systeme-triple-orthogonal.html?lang=fr

GASPARD MONGE
Le mémoire sur les déblais et les remblais
Le *20 janvier 2012* - Ecrit par [Étienne Ghys]
https://images.math.cnrs.fr/Gaspard-Monge,1094.html?lang=fr

Obrigado!!