Limit cycles for fewnomial differential equations

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Memorial day in honor of Jorge Sotomayor

March 25th, 2022

1. The influence of Jorge Sotomayor on my research
    - Joint works
    - His influence goes beyond
2. Fewnomial differential equations
    - Main results
    - Case with 3 monomials
    - Case with 2 monomials
    - Future work

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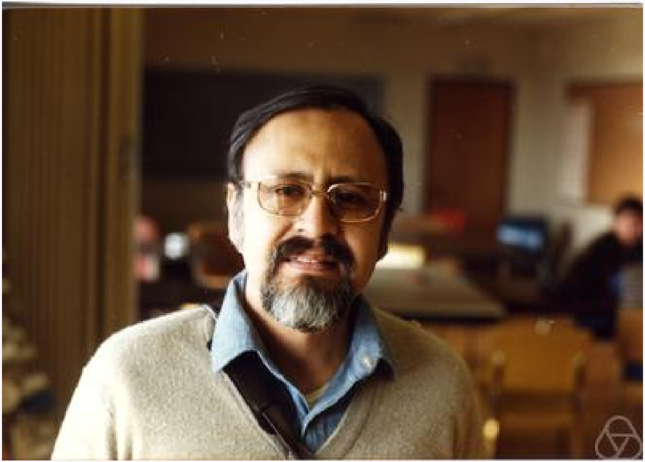
Sotomayor at Oberwolfach in 1983



Joint publications

- A. G., J. Llibre, J. Sotomayor. Limit cycles of vector fields of the form $X(v) = Av + f(v)Bv$. J. Differential Equations, **67**, 90–110. 1987.
- A. G., J. Llibre, J. Sotomayor. Further considerations on the number of limit cycles of vector fields of the form $X(v) = Av + f(v)Bv$. J. Differential Equations, **68**, 36–40. 1987.
- A. G., J. Sotomayor. On the basin of attraction of dissipative planar vector fields. In Bifurcations of planar vector fields (Luminy, 1989), 187–195. Springer, Berlin, 1990.
- A. G., J. Llibre, J. Sotomayor. Global asymptotic stability of differential equations in the plane. J. Differential Equations, **91**, 327–335. 1991.

Influence of our joint work on my research-I

- A. G., J. Llibre, J. Sotomayor. Limit cycles of vector fields of the form $X(v) = Av + f(v)Bv$. J. Differential Equations, **67**, 90–110. 1987.
- A. G., J. Llibre, J. Sotomayor. Further considerations on the number of limit cycles of vector fields of the form $X(v) = Av + f(v)Bv$. J. Differential Equations, **68**, 36–40. 1987.

LINES OF INTEREST:

- Bifurcations
- Number of limit cycles
- Cherkas transformation and Abel differential equations

Influence of our joint work on my research-II

A. G., J. Sotomayor. On the basin of attraction of dissipative planar vector fields. In Bifurcations of planar vector fields (Luminy, 1989), 187–195. Springer, Berlin, 1990.

LINES OF INTEREST:

- Global attraction
- Size of the basin of attraction. Control Theory.

For instance, the title of my most recent preprint is: *Asymptotic stability for block triangular maps*, A. Cima, A. G. and V. Mañosa.

Influence of our joint work on my research-III

A. G., J. LLIBRE, J. SOTOMAYOR. Global asymptotic stability of differential equations in the plane. J. Differential Equations, **91**, 327–335. 1991.

LINES OF INTEREST:

- Markus-Yamabe and LaSalle type problems for continuous and discrete dynamical systems
- Jacobian Conjecture

His role was fundamental in the solution of the Markus-Yamabe conjecture on the plane. He arrived to Barcelona with the preprint:

G. H. MEISTERS, C. OLECH. Solutions of the global stability Jacobian conjecture for the polynomial case, Analyse Mathématique et applications, Gauthier-Villars, Paris, 373–381, 1988.

From that work we wrote our joint paper and it motivated Carlos Gutiérrez (and also to other authors) to study it. In fact, after my talk in the interesting meeting at IMPA about this subject:

INTERNATIONAL CONFERENCE AND WORKSHOP ON DYNAMICAL SYSTEMS

DE 31 DE JULHO A 18 DE AGOSTO DE 1989

IMPA - RIO DE JANEIRO

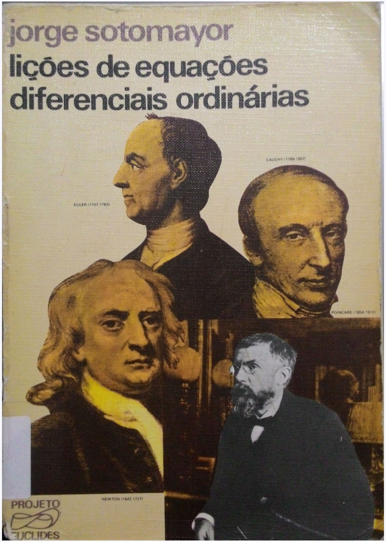
Carlos Gutierrez came to us and ask for more details about our work, because he told us that he had some idea to solve the conjecture.

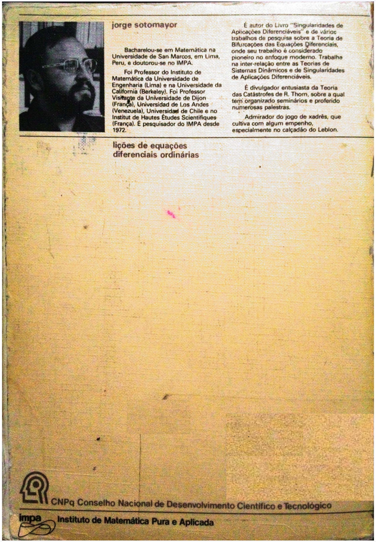
His influence goes beyond !

- Most people in Barcelona has learned non-linear ODE studying his book that we simply call Sotomayor while many post graduate students have learned bifurcations theory and the Poincaré compactification studying his notes that we lovingly call Sotomenor[1].

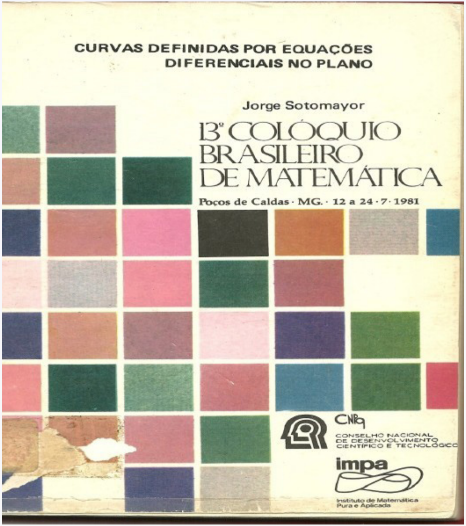
[1] Here we play with the meaning in Spanish of:

"mayor" $\rightarrow$ bigger, "menor" $\rightarrow$ lesser


jorge sotomayor
lições de equações
diferenciais ordinárias
PROJETO
EUCLIDES


jorge sotomayor
Bacharelou-se em Matemática na Universidade de San Marcos, em Lima, Peru, e doutorou-se no IMPA.
Foi Professor do Instituto de Matemática da Universidade de Engenharia (Lima) e na Universidade da California (Berkeley). Foi Professor Visitante da Universidade de Dijon (França), Universidad de Los Andes (Venezuela), Universidad de Chile e no Institut de Hautes Études Scientifiques (França). É pesquisador do IMPA desde 1972.
É autor do Livro "Singularidades de Aplicações Diferenciáveis" e de vários trabalhos de pesquisa sobre a Teoria de Bifurcações das Equações Diferenciais, onde seu trabalho é considerado pioneiro no enfoque moderno. Trabalha na inter-relação entre as Teorias de Sistemas Dinâmicos e de Singularidades de Aplicações Diferenciáveis.
É divulgador entusiasta da Teoria das Catástrofes de R. Thom, sobre a qual tem organizado seminários e proferido numerosas palestras.
Admirador do jogo de xadrês, que cultiva com algum empenho, especialmente no calçadão do Leblon.
lições de equações
diferenciais ordinárias
CNPq Conselho Nacional de Desenvolvimento Científico e Tecnológico
impa Instituto de Matemática Pura e Aplicada

Sotomenor (IMPA, 1981)



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ÍNDICE

His influence goes beyond !

- I did my first research stage in IMPA, for two months in 1989. There, apart from mathematics, I learnt that sometimes the research relations go beyond and are transformed into friendship relations. My wife and I even lived for some days in Marilda and Soto's guest house at Rio.
- He motivates me to study in detail the behaviour of the period function. He, together with C. Chicone where the impulsors of that line of research around the world.
- When my university (UAB) founded the journal Materials Matemàtics (Mat^2) devoted to popularize mathematics, he immediately collaborate publishing in 2007 the interesting paper:

El elipsoide de Monge y las líneas de curvatura

His influence goes beyond !

MATerials MATemàtics
Volum 2007, treball no. 1, 25 pp. ISSN: 1887-1097
Publicació electrònica de divulgació del Departament de Matemàtiques
de la Universitat Autònoma de Barcelona
www.mat.uab.cat/matmat

El elipsoide de Monge y las líneas de curvatura*

Jorge Sotomayor**

1. El Elipsoide

Esta historia se inicia en una calurosa noche de octubre de 1970 en Río de Janeiro. Víctima del insomnio, decidí fisgonear los libros que mi esposa había acomodado cuidadosamente en nuestro estante. Recientemente ella había colocado allí un buen número de libros suyos.

Mi cándida y reposada actitud contrastaba con una extraña tensión que emanaba del estante, inundando la sala. Intrigado me vi impelido a averiguar la causa. Arrinconado en una esquina, envuelto por una elegante pasta verde, pulsaba inquieto el libro de Struik "*Lecciones de Geometría Diferencial Clásica*", Aguilar 1955.

Mi natural atracción por la Geometría y por la lengua de Cervantes me impulsaron, ingenuamente, a abrirlo. Y lo hice justamente en una página de la cual, como si hubiera estado al acecho, surgió la figura del elipsoide triaxial que ilustra esta página.

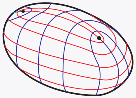


Figura 1: El Elipsoide de Monge

*Traducción libre, adaptación y actualización del artículo publicado en Portugués en *Matemática Universitária*, SBM, 15, 1993.

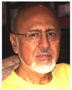
**El autor contó con el apoyo parcial del CNPq (Conselho Nacional de Desenvolvimiento Científico e Tecnológico), Brasil.

B. Smyth, *The nature of elliptic sectors in the principal foliations of surface theory*, EQUADIFF 2003, 957–959, World Sci. Publ., Hackensack, NJ, 2005.

B. Smyth y F. Xavier, *A sharp geometric estimate for the index of an umbilic on a smooth surface*, Bull. London Math. Soc. **24**, 1992, no. 2, 176–180.

B. Smyth y F. Xavier, *Eigenvalue estimates and the index of Hessian fields*, Bull. London Math. Soc. **33**, 2001, no. 1, 109–112.

Agradecimientos. Dejamos constancia de nuestra gratitud a Ronaldo Garcia, Luis Fernando Mello y Tiago de Carvalho por sus observaciones y comentarios al texto actual (los dos primeros) y por su ayuda en la preparación de las figuras (el primero y el último).



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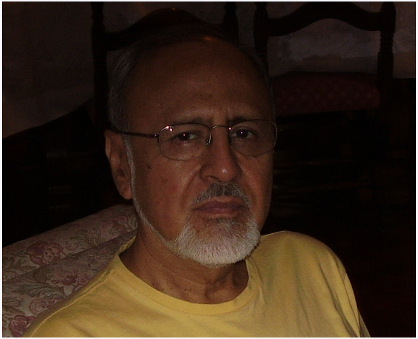
Publicat el 24 de gener de 2007

His influence goes beyond !



- He, together with M. A. Teixeira and J. Llibre were the impulsors of the fruitful relations between the Dynamical systems grup in Barcelona and several groups in Brazil.

His influence goes beyond !

For instance, I have been collaborating with:

- R. A. Garcia, A. G., A. Guillamon. Geometrical conditions for the stability of orbits in planar systems. Math. Proc. Cambridge Philos. Soc., **120**, 499–519. 1996.
- A. Cima, A. G., P. R. Silva. On the number of critical periods for planar polynomial systems. Nonlinear Anal., **69**, 1889–1903. 2008.
- A. Cima, A. G., J. C. Medrado. On persistent centers. Bull. Sci. Math., **133**, 644–657. 2009.
- C. A. Buzzi, A. G., J. Torregrosa. Algebraic limit cycles in piecewise linear differential systems. Internat. J. Bifur. Chaos Appl. Sci. Engrg., **28**, 1850039 (14 pages). 2018.
- C. A. Buzzi, Y. R. Carvalho, A. G.. The local period function for Hamiltonian systems with applications. J. Differential Equations, **280**, 590–617. 2021.

Barcelona-Brasil, November 2004 at CRM

His influence goes beyond !

Around 2000, we discussed in several occasions about the number of limit cycles of planar linear piecewise differential equations, with a line of discontinuity. Very soon he gave me very interesting hand written notes with examples with **TWO LIMIT CYCLES**.

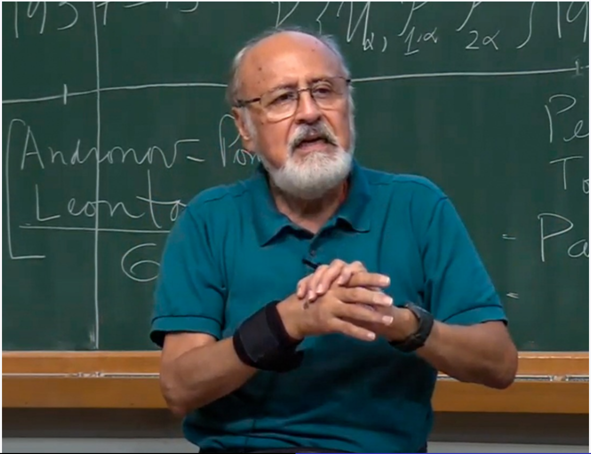
At that time this was unknown and I think that a very interesting result. Unfortunately, other projects made that we never wrote a paper with these results.

I have thought several times that this has been one of the faults of my carrier.

His influence goes beyond !

An advice:

> *If an outstanding mathematician like SOTO gives you some indications you MUST continue thinking in that direction!*



His influence goes beyond !

By the way, his proof got the 2 limit cycles from the bifurcation of a heteroclinic connection of 2 linear saddles, each one of them in each of the two half planes.

Unfortunately I have not been able to find these notes.

Nowadays it is known that there are examples with **THREE LIMIT CYCLES** and this is a conjecture. See for instance,

J. Llibre, E. Ponce. Three nested limit cycles in discontinuous piecewise linear differential systems with two zones. Dynam. Contin. Discrete Impuls. Systems. Ser. B, **19**, 325–335. 2012.

and its references.

His influence goes beyond !

During these last years we keep the contact by mail. I frequently received his interesting messages with several papers and news.

When I finished some work, usually I sent it to Soto. Often I received some very interesting comments on it.

For instance, when I send it my recent work *Some open problems in low dimensional dynamical systems* published in SeMA J. 2021, he gave me several hints of how to improve the presentation of the mathematical questions.

Some weeks ago I ended a work that I thought that could interest him and my first thought was: "I have to send it to Soto"

JORGE, we will miss you!
!JORGE, te echaremos mucho de menos!

1. The influence of Jorge Sotomayor on my research
    - Joint works
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2. Fewnomial differential equations
    - Main results
    - Case with 3 monomials
    - Case with 2 monomials
    - Future work

The problem of my talk: limit cycles for fewnomial ODE

It is well known that

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \quad (x,y) \in \mathbb{R}^2, \quad t \in \mathbb{R},$$

with P and Q polynomials can also be written as

$$\frac{dz}{dt} = \dot{z} = F(z,\bar{z}), \quad z \in \mathbb{C}, \quad t \in \mathbb{R},$$

where F is a complex polynomial.
We consider systems with F having few monomials and study the number of limit cycles of them. We will call them fewnomial differential equations.

Talk based on the papers:

- A. G., Chengzhi Li, J. Torregrosa. Limit cycles for 3-monomial differential equations. J. Math. Anal. Appl., **428**, 735–749. 2015.
- M. J. Álvarez, A. G., R. Prohens. Uniqueness of limit cycles for complex differential equations with two monomials. Preprint, March 2022.

Main result

Theorem

(i) *The differential equations*

$$\dot{z} = az^k\bar{z}^l$$

have NO limit cycle.

(i) *The differential equations*

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n,$$

have at most ONE limit cycle. Moreover, this upper bound is sharp and when it exists it is hyperbolic.

(iii) *The differential equations*

$$\dot{z} = az^u\bar{z}^v + bz^k\bar{z}^l + cz^m\bar{z}^n,$$

have NO UPPER BOUND for their number of limit cycles.

Proof of item (i): It is well-known

Clearly, equations with one monomial

$$\dot{z} = az^k\bar{z}^l$$

have NO limit cycle because they are homogeneous.

In fact, it is well-known that planar homogeneous vector fields can have periodic orbits (and then they have a global center) but never have limit cycles.

Proof of item (iii): case with 3 monomials

The result is a consequence of the following more concrete result, that is proved in the first quoted paper:

Theorem

For any $p \in \mathbb{N}$ there is a differential equation of type

$$\dot{z} = az + bz^k\bar{z}^l + cz^m\bar{z}^n,$$

with $a, b, c \in \mathbb{C}$ and $k, l, m, n \in \mathbb{N} \cup \{0\}$, having at least p limit cycles.

Let us give some ideas about the proof.

A well-known example with 3 monomials

A celebrated family of differential equations with 3 monomials is

$$\dot{z} = az + bz^2\bar{z} + c\bar{z}^{q-1},$$

with $q \geq 3$. It gives the *versal deformation* of a principal singular smooth systems having rotational invariance of $2\pi/q$ radians. The cases $q = 3, 4$ are called *strong resonances* while the cases $q \geq 5$ are called *weak resonances*. The situation $q \neq 4$ is well understood, see for instance:

- ARNOLD, V., "Chapitres supplémentaires de la théorie des équations différentielles ordinaires", Ed. Mir-Moscou, 1980.
- CHOW, S. N.; LI, C.; WANG, D., *A simple proof of the uniqueness of periodic orbits in the 1:3 resonance problem.* Proc. Amer. Math. Soc. **105** (1989) 1025–1032.
- HOROZOV, E., *Versal deformations of equivariant vector fields for the case of symmetries of order 2 and 3*. Trudy Sem. Pet., **5** (1979) 163–192.

A well-known example with 3 monomials-II

The study of the limit cycles for case $q = 4$ for

$$\dot{z} = az + bz^2\bar{z} + c\bar{z}^{q-1},$$

turns out to be more difficult problem. To know the number of limit cycles surrounding the origin, and eventually surrounding also the other 4 or 8 critical point that the equation can posses is yet an open question. It is known that:

- It has at least 2 limit cycles surrounding the 9 critical points.
- The problem of the number of limit cycles not surrounding the origin was solved by Zegeling. There are either NO limit cycle or exactly 4 hyperbolic ones, each one of them surrounding exactly one of the critical points of index $+1$.

Inspired by the presence of the four limit cycles, each one surrounding a different critical point, we want to do something similar but with any number of critical points.

Starting equation for proving item (iii)

It can be proved that the differential equations

$$\dot{z} = az + bz^2\bar{z} + c\bar{z}^{q-1}, \tag{1}$$

for $q \geq 5$ have only one critical point of index $+1$.
Therefore, instead of considering them we take the following subclass of equations with 3 monomials,

$$\dot{z} = a\,z + b\,z^{p-1}\bar{z}^{p-2} + c\,\bar{z}^{p-1} = a\,z + b\,z|z|^{2(p-2)} + c\,\bar{z}^{p-1}, \tag{2}$$

with $p \geq 3$, which also have rotational invariance of $2\pi/p$ radians.
Notice that both coincide when $p = q = 3$.

Starting equation for proving for proving item (iii)-III

Equation

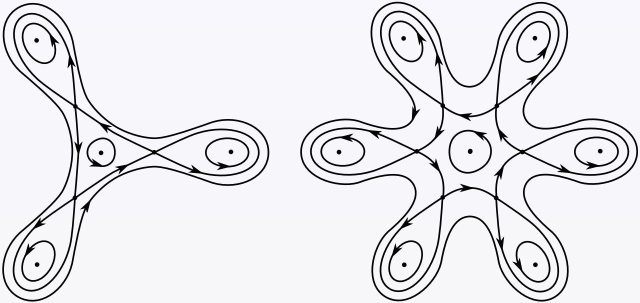
$$\dot{z} = (a_1 + i)\, z + (b_1 + i)\, z^{p-1} \bar{z}^{p-2} - \frac{5i}{2}\, \bar{z}^{p-1},$$

when $a_1 = b_1 = 0$ is Hamiltonian, with Hamiltonian function

$$H(r, \theta) = \frac{r^2}{2} - \frac{5}{2p} r^p \cos(p\,\theta) + \frac{r^{2(p-1)}}{2(p-1)} - \tilde{\rho},$$

where $\tilde{\rho} = \frac{(p-2)(p-5)}{2p(p-1)}\, 2^{\frac{2}{p-2}}$.

Their phase portraits are:



Centers when $a_1 = b_1 = 0$ for the cases $p = 3$ and $p = 6$.

Proof of item (iii)

It is a corollary of the following proposition:

Proposition

For $3 \le p \in \mathbb{N}$, consider the 2-parameter family of systems

$$\dot{z} = (a_1 + i)\, z + (b_1 + i)\, z|z|^{2(p-2)} - \frac{5i}{2}\, \bar{z}^{p-1},$$

with $a_1, b_1 \in \mathbb{R}$, $3 \le p \in \mathbb{N}$. Then there exist values for a_1 and b_1 for which the above equation has at least p limit cycles.

Proof of the proposition

The differential equation in polar coordinates is

$$dH(r,\theta) - (a_1\, r^2 + b_1\, r^{2(p-1)})\, d\theta = 0.$$

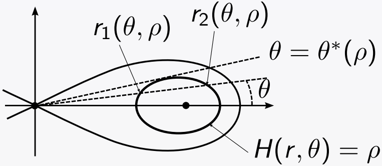
Writing $a_1 = \varepsilon\,\alpha$ and $b_1 = \varepsilon\,\beta$, for $\alpha, \beta \in \mathbb{R}$ and ε small enough, the associated first order Melnikov function is

$$M(\rho) = \alpha\, I_2(\rho) + \beta\, I_{2(p-1)}(\rho),$$

where

$$I_j(\rho) = \int_{H=\rho} r^j\, d\theta = 2\int_0^{\theta^*(\rho)} \left(r_2^j(\theta,\rho) - r_1^j(\theta,\rho) \right)\, d\theta,$$

for $j = 2, 2(p-1)$ and $\rho \in (\rho^*, 0)$.



Proof of the proposition-II

Then, we introduce the auxiliary analytic function

$$J(\rho) = \frac{I_{2(p-1)}(\rho)}{I_2(\rho)}, \quad \rho \in (\rho^*, 0)$$

and we write

$$M(\rho) = I_2(\rho)\big(\alpha + \beta J(\rho)\big).$$

Notice that $I_2(\rho) > 0$ because this function gives the double of the area surrounded by a connected component of the curve $H(r, \theta) = \rho$.
Then, by proving that $J(\rho)$ is not constant the proof follows by using the usual methods. We omit this proof, that is quite technical. See the quoted paper for more details

Proof of item (ii): the 2-monomials case

Theorem

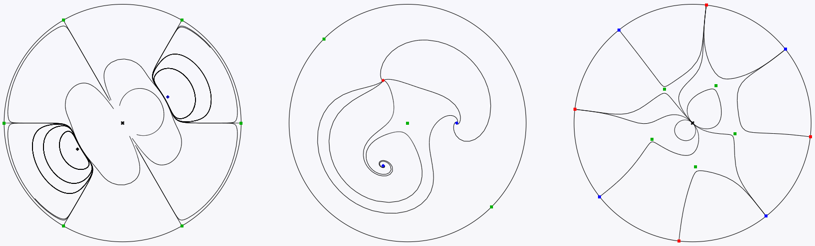
Differential equation

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n, \ z \in \mathbb{C},$$

with $k, l, m, n \in \mathbb{Z}^+ \cup \{0\}$, *and* $a, b, z \in \mathbb{C}$ *has at most one limit cycle and it exists if and only if* $k - l = m - n = 1$, $\operatorname{Re}(a)\operatorname{Re}(b) < 0$ *and* $a/b \notin \mathbb{R}^-$ *and it is the circle* $x^2 + y^2 = (-\operatorname{Re}(a)/\operatorname{Re}(b))^{n-l}$. *Moreover it is hyperbolic.*

Main difficulty: to prove that in most cases there is NO limit cycle.

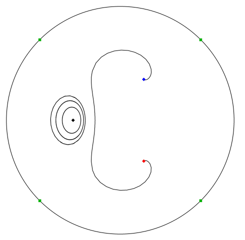
There is a huge variety of phase portraits. Done with P4.



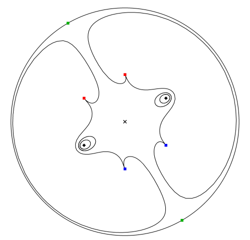
(a) (b) (c)

Proof of item (ii): the 2-monomials case

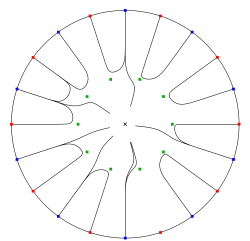
There is a huge variety of phase portraits. Done with P4.



(a)



(b)



(c)

Proof of item (ii): several facts

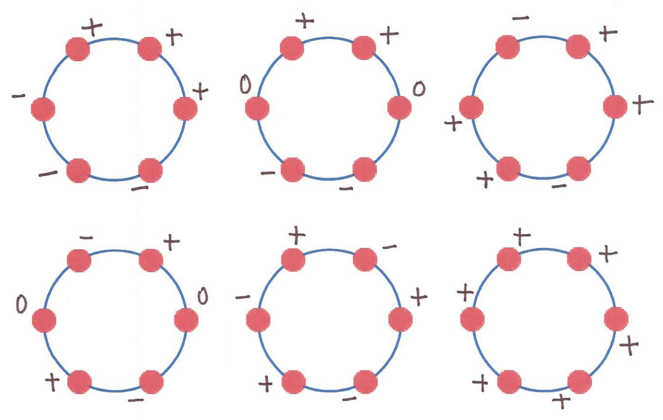
To have an idea of the situation we list some properties that we have found:

- When $k + l \neq 0$ the origin is a critical point of index $k - l$.
- When $q = l - k + m - n \neq 0$, apart from the origin, the differential equation has exactly $|q|$ simple critical points, all them located on a circle $\mathbb{S}^1$ centered at the origin. Moreover, when $q > 0$ (resp. $q < 0$) all are anti-saddles (res. saddles).
- On the one hand, all nonzero weak foci are centers. On the other, the origin can be a weak-focus that is not a center. Moreover in this case a limit cycle can bifurcate from it via a Andronov-Hopf type bifurcation.
- The center-focus problem can be solved and all centers are reversible.

Proof of item (ii): several facts

- As a consequence of Lagrange's theorem about the cardinality of the subgroups of finite groups we prove that the number of nonzero centers is a divisor of $q > 0$.
- There is NO upper bound for the number of centers that the family of differential equations can have.
- The origin and the infinity (in the compactification of the differential equation to $\mathbb{S}^2$ by adding one point), can be sometimes simultaneously centers.
- A limit cycles only exists when $k - l = m - n = 1$ and the origin is the unique critical point.
- When the differential equation has $|q|$ nonzero anti-saddles we prove a Berlinskiĭ type results concerning the geometrical distribution of their stabilities.

Berlinskiĭ type results: case $q = 6$



$$\text{attractor} \longrightarrow -, \quad \text{repellor} \longrightarrow +, \quad \text{center} \longrightarrow 0$$

All singularities of center type also happens.
All $+$ and $-$ symbols can be also interchanged.

Proof of item (ii): some tools

Non-existence of limit cycles surrounding the origin.

Proposition

Consider the differential equation

$$\dot{z} = X_N(z,\bar{z}) + X_M(z,\bar{z}),\ N < M,$$

where, X_j is a homogeneous vector field of degree j in the variables z and $\bar{z}$. If one of the following conditions hold:

(i) *The differential equation $\dot{z} = X_N(z,\bar{z})$ has an invariant straight line through the origin and M is even,*

(ii) *The differential equation $\dot{z} = X_M(z,\bar{z})$ has an invariant straight line through the origin and N is even,*

then it has not periodic orbits surrounding the origin.

The above result extends the fact that limit cycles never surround a node for QS.

Proof of item (ii): some tools

Non-existence of limit cycles surrounding a non-zero critical point.

Lemma

If our differential equation with 2 monomials has a nonzero critical point, after a linear change of coordinates and positive constant rescaling of time it can be written as:

$$\dot{z} = c(z^k\bar{z}^l - z^m\bar{z}^n), \quad \textit{where} \quad c = e^{i\alpha}. \tag{3}$$

Proposition

Let the origin be a center for a smooth differential equation $\dot{z} = iF(z,\bar{z})$ and let $\mathcal{U}$ be its period annulus. Then for $\delta \notin \{\pi/2, -\pi/2\}$ the differential equation $\dot{z} = e^{i\delta}F(z,\bar{z})$ has not periodic orbits intersecting the set $\mathcal{U}$. Moreover, if F is analytic it has not periodic orbits surrounding only the origin.

The proof uses rotated families of vector fields.

Future work

We continue working on the following questions:
Consider

$$\dot{z} = az^u\bar{z}^v + bz^k\bar{z}^l + cz^m\bar{z}^n, \qquad (4)$$

Define $\mathcal{H}_3(N)$ as the supremum of the number of limit cycles of (4) when $\max(u+v, k+l, m+n) = N$ and $a, b, c \in \mathbb{C}$.

PROBLEMS:

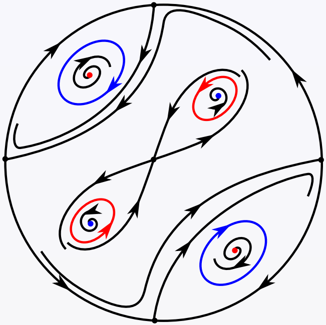
- Give a direct proof that the number of limit cycles of each differential equation (4) is finite.
- Prove that $\mathcal{H}_3(N)$ is finite.
- Give upper and lower bounds of $\mathcal{H}_3(N)$.

Example showing that $\mathcal{H}_3(3) \geq 4$

Our numerical simulations also show that differential equation

$$\dot{z} = (1 + 9i/4)\, z + (13/4 + i/2)\, \bar{z} + z^3,$$

has at least four limit cycles, see its phase portrait on the Poincaré disc.



The classical example with rotational symmetry and $q = 4$ also shows that $\mathcal{H}_3(3) \geq 4$.

Thank you very much for your attention