Limit cycles of generic piecewise center-type vector fields in $\mathbb{R}^3$ separated by either one plane or by two parallel planes

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Abstract

While the limit cycles of the piecewise differential systems in the plane $\mathbb{R}^2$ have been studied intensively during these last twenty years, this is not the case for the limit cycles of the piecewise differential systems in the space $\mathbb{R}^3$.

The goal of this article is to study the continuous and discontinuous piecewise differential systems in $\mathbb{R}^3$, formed by linear vector fields similar to planar centers separated by one or two parallel planes. We call those "center-type" differential systems, which have two pure imaginary numbers and zero as eigenvalues. When these kinds of piecewise differential systems are continuous or discontinuous separated by one plane, then they have no limit cycles. Also, if they are continuous separated by two planes, then generically they do not have limit cycles. But when the piecewise differential systems are discontinuous separated two parallel planes, we show that generically they can have at most four limit cycles, and that there exist such systems with four limit cycles. The genericity here means that the statements hold in a residual set of the space of parameters associated to the differential system.

We recall that the same problem but for discontinuous piecewise differential systems in $\mathbb{R}^2$ formed by linear differential centers separated by two parallel

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straight lines have at most one limit cycle.

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1. Introduction and statement of the main results

The study of piecewise vector fields (PVF) goes back to Andronov, Vitt, and Khaikin [1] and still continues to receive strong attention from researchers. These last years a renewed interest has appeared in the mathematical community working in differential equations for understanding the dynamical richness of the piecewise vector fields, because these vector fields are widely used to model processes appearing in electronics, mechanics, economy, etc., see for instance the books [2] and [3] and the survey of [4] and the hundreds of references quoted in these last three works.

In this paper, we shall work with piecewise vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$, and the definition of these vector fields on the separation line of their pieces in $\mathbb{R}^2$, or on the discontinuity region of their pieces in $\mathbb{R}^3$ follow the rules of Filippov [5], see a summary of these rules in Section 2.

These last two decades the limit cycles of the piecewise differential systems in the plane $\mathbb{R}^2$ have been studied intensively, see for instance [6–30]. This is not the case for the limit cycles of the piecewise differential systems in the space $\mathbb{R}^3$.

A *center* of a differential system in the plane $\mathbb{R}^2$ is an equilibrium point $\mathbf{x}$ having a neighbourhood U such that $U \setminus \{\mathbf{x}\}$ is filled of periodic orbits. A *global* center is a center $\mathbf{x}$ such that $\mathbb{R}^2 \setminus \{\mathbf{x}\}$ is filled of periodic orbits. The notion of a center appeared already in the works of Poincaré [31] in 1881 and Dulac [32] in 1908.

One of the main objects in the study of the vector fields is the limit cycles. A *limit cycle* of a vector field is a periodic orbit isolated in the set of all periodic orbits of the vector field.

In the paper [28] it is proved that continuous and discontinuous piecewise vector fields in $\mathbb{R}^2$ formed by two pieces separated by one straight line and formed by two arbitrary linear centers cannot have limit cycles, and that also the continuous piecewise vector fields in $\mathbb{R}^2$ formed by three pieces separated by two parallel straight lines and formed by three arbitrary linear centers have no limit cycles. But the discontinuous piecewise differential systems separated by two parallel straight lines and formed by three arbitrary linear centers can have at most one limit cycle, and there are such systems which one limit cycle. The objective of this paper is to study a similar problem for continuous and discontinuous piecewise vector fields in $\mathbb{R}^3$.

In $\mathbb{R}^3$ there are no centers in the sense that there are no equilibrium points $\mathbf{x}$ having a neighborhood U such that $U \setminus \{\mathbf{x}\}$ is filled with periodic orbits, see for instance [33].

The main goal of this paper is to study the limit cycles of the continuous and discontinuous piecewise vector fields in $\mathbb{R}^3$ separated by one plane or by two parallel planes and formed by linear "center-type" vector fields of $\mathbb{R}^3$.

More precisely, we consider the linear vector field

$$X(\mathbf{x}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = A\mathbf{x} + B, \tag{1}$$

where

$$\begin{aligned}
A_{11} &= a_1a_3c_2 + b_1b_3c_2 - a_1a_2c_3 - b_1b_2c_3, \\
A_{12} &= a_2a_3c_2 + b_2b_3c_2 - a_2^2c_3 - b_2^2c_3, \\
A_{13} &= a_3^2c_2 + b_3^2c_2 - a_2a_3c_3 - b_2b_3c_3, \\
A_{21} &= -a_1a_3c_1 - b_1b_3c_1 + a_1^2c_3 + b_1^2c_3, \\
A_{22} &= -a_2a_3c_1 - b_2b_3c_1 + a_1a_2c_3 + b_1b_2c_3, \\
A_{23} &= -a_3^2c_1 - b_3^2c_1 + a_1a_3c_3 + b_1b_3c_3, \\
A_{31} &= a_1a_2c_1 + b_1b_2c_1 - a_1^2c_2 - b_1^2c_2, \\
A_{32} &= a_2^2c_1 + b_2^2c_1 - a_1a_2c_2 - b_1b_2c_2, \\
A_{33} &= a_2a_3c_1 + b_2b_3c_1 - a_1a_3c_2 - b_1b_3c_2,
\end{aligned}$$

$$B_1 = a_3a_4c_2 + b_3b_4c_2 - a_2a_4c_3 - b_2b_4c_3,$$
$$B_2 = -a_3a_4c_1 - b_3b_4c_1 + a_1a_4c_3 + b_1b_4c_3,$$
$$B_3 = a_2a_4c_1 + b_2b_4c_1 - a_1a_4c_2 - b_1b_4c_2.$$

The vector field (1) is obtained after applying the affine transformation

$$(x, y, z) \mapsto (a_1x + a_2y + a_3z + a_4, b_1x + b_2y + b_3z + b_4, c_1x + c_2y + c_3z + c_4)$$

with inverse

$$\begin{aligned}(x, y, z) \quad \mapsto \quad & \tfrac{-1}{det(A)}(a_4b_3c_2 - a_3b_4c_2 - a_4b_2c_3 + a_2b_4c_3 + a_3b_2c_4 - a_2b_3c_4 \\ & -b_3c_2x + b_2c_3x + a_3c_2y - a_2c_3y - a_3b_2z + a_2b_3z, -a_4b_3c_1 \\ & +a_3b_4c_1 + a_4b_1c_3 - a_1b_4c_3 - a_3b_1c_4 + a_1b_3c_4 + b_3c_1x - b_1c_3x \\ & -a_3c_1y + a_1c_3y + a_3b_1z - a_1b_3z, a_4b_2c_1 - a_2b_4c_1 - a_4b_1c_2 \\ & +a_1b_4c_2 + a_2b_1c_4 - a_1b_2c_4 - b_2c_1x + b_1c_2x + a_2c_1y - a_1c_2y \\ & -a_2b_1z + a_1b_2z).\end{aligned}$$

to the linear differential system

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{z} = 0, \tag{2}$$

which has the two independent first integrals $f_1(\mathbf{x}) = x^2 + y^2$ and $f_2(\mathbf{x}) = z$. Here we assume that $det(A) = -a_3b_2c_1 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 + a_1b_2c_3 \neq 0$. We emphasize the genericity in this paper means that some result holds in a residual set.

Note that the linear differential system (2) in $\mathbb{R}^3$ has the full z-axis filled with singular points, and on the invariant planes $z = z_0$ =constant we have a global center at $(0, 0, z_0)$, i.e. all the periodic orbits surrounding the center $(0, 0, z_0)$ in the plane $z = z_0$ fill this plane with the exception of the singular point $(0, 0, z_0)$.

In this paper the linear differential system in $\mathbb{R}^3$ defined by a vector fields X will be called a *linear center* or simply a *center* in $\mathbb{R}^3$.

Changing the parameters (a_i, b_i, c_i) of the vector field $X(\mathbf{x})$ to $(\alpha_i, \beta_i, \gamma_i)$ for $i = 1, 2, 3$ we get another linear vector field $Y(\mathbf{x}) = C\mathbf{x} + D$.

For any piecewise vector field in $\mathbb{R}^3$ formed by two pieces and separated by one plane we can assume without loss of generality, after an affine change in $\mathbb{R}^2$ if necessary, that such a plane is the plane $x = 0$.

We define the *piecewise vector field* $N = (X, Y)$ *in* $\mathbb{R}^3$ *of two pieces separated by the plane* $x = 0$ *and formed by the two centers* X *and* Y *of* $\mathbb{R}^3$ as

$$N(\mathbf{x}) = \begin{cases} X(\mathbf{x}) = A\mathbf{x} + B, \text{ if } x \geq 0, \\ Y(\mathbf{x}) = C\mathbf{x} + D, \text{ if } x \leq 0. \end{cases}$$

If $X = Y$ on the plane $x = 0$ we say that the piecewise vector field N is *continuous*, and if $X \neq Y$ on the plane $x = 0$ we say that the piecewise vector field N is *discontinuous*. For these two kinds of piecewise vector fields N we have the following two results.

Theorem 1.1. *A piecewise vector field in* $\mathbb{R}^3$ *separated by one plane and formed by two linear centers* X *and* Y *has no limit cycles.*

The limit cycles of Theorem 1.1 are crossing limit cycles, see for details the last part of Section 2.

If we assume that the discontinuity region of a piecewise vector field in $\mathbb{R}^3$ is formed by two parallel planes, then without loss of generality we can assume, after an affine change in $\mathbb{R}^3$ if necessary, that these parallel planes are the planes $x = 1$ and $x = -1$.

We consider piecewise vector fields separated by the planes $x = \pm 1$ and formed by three centers X, Y and Z, where $Z(\mathbf{x}) = E\mathbf{x} + F$ is obtained from X changing the coefficients (a_i, b_i, c_i) by (A_i, B_i, C_i) for $i = 1, 2, 3$.

More precisely, we define the *piecewise vector field* $M = (X, Y, Z)$ *in* $\mathbb{R}^3$ *of three pieces separated by the two planes* $x = \pm 1$ *and formed by the three centers* X, Y *and* Z *of* $\mathbb{R}^3$ as

$$M(\mathbf{x}) = \begin{cases} X(\mathbf{x}) = A\mathbf{x} + B, \text{ if } x \geq 1, \\ Y(\mathbf{x}) = C\mathbf{x} + D, \text{ if } -1 \leq x \leq 1, \\ Z(\mathbf{x}) = E\mathbf{x} + F, \text{ if } x \leq -1. \end{cases}$$

As in the case when the discontinuity line was formed by a unique plane, now we can consider continuous and discontinuous piecewise vector fields. Our main results for the piecewise vector fields M are the following two theorems.

Theorem 1.2. *A continuous piecewise vector field separated by two parallel planes and formed by three linear centers X, Y and Z generically has no limit cycles.*

We notice that our parameter space is $\mathbb{R}^{36}$. Thus, when we claim that the continuous piecewise vector field has generically no limit cycle we mean that it has no limit cycles in a residual set of $\mathbb{R}^{36}$. It other others, the possible scenario where the result could not be verified has measure zero (here we can assume Lebesgue measure).

Theorem 1.3. *A discontinuous piecewise vector field separated by two parallel planes and formed by three linear centers X, Y and Z generically has at most four limit cycles, and we provide one of these piecewise vector fields with exactly four limit cycles.*

The meaning of genericity in the statement of Theorem 1.3 is the same as in Theorem 1.2. Again the limit cycles of Theorems 1.2 and 1.3 are crossing limit cycles, see for details the last part of Section 2. The proofs of Theorems 1.1, 1.2 and 1.3 are given in the Section 3.

2. Filippov rules for defining the piecewise vector fields

Following the Filippov rules introduced in [5] we first consider an open set $U \subset \mathbb{R}^3$ and the *discontinuity region* $\Sigma = f^{-1}(0)$ being 0 a regular value of a C^r smooth function $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, for $1 \leq r \leq \infty$. As usual a C^r vector field is a C^r function $X : U \to \mathbb{R}^3$. The set of C^r vector fields over $\mathbb{R}^3$ will denoted by $\mathfrak{X}^r(\mathbb{R}^3)$.

Let $X_i \in \mathfrak{X}^r(\mathbb{R}^3)$ be arbitrary vector fields, with $i = 1, \ldots, n$ where n is the number of connected components D_i of $\mathbb{R}^3 \setminus \Sigma$. We denote a (non-smooth) piecewise vector field in $\mathbb{R}^3$ by the n-tuple $N = (X_1, \ldots, X_n)$ where

$$N(\mathbf{x}) = X_i(\mathbf{x}), \quad \text{if}\, \mathbf{x} \,\text{in}\, D_i,$$

here $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. We note that on the discontinuity region Σ the piecewise vector field N is bi-valuated.

Now we precisely define $N = (X, Y)$ over Σ. A point $\mathbf{x} \in \Sigma$ is of *crossing* type if vector fields $X(\mathbf{x})$ and $Y(\mathbf{x})$ points in the same direction respect to Σ. It is of *sliding* type if both $X(\mathbf{x})$ and $Y(\mathbf{x})$ points inward Σ and it is of *escaping* type if $X(\mathbf{x})$ and $Y(\mathbf{x})$ points outward Σ. In each situation we are assuming that the trajectories of X and Y are transversal to Σ. Otherwise we say that a point $\mathbf{x} \in \Sigma$ is a *tangency point* of X or Y.

An effective criterion for classifying points on the discontinuity region Σ can be established in terms of the Lie derivatives as follows. We define the Lie derivative at $\mathbf{x} \in \Sigma$ as

$$Xf(\mathbf{x}) = \langle \nabla f(\mathbf{x}), X(\mathbf{x}), \rangle$$

and for $k \geq 2$ we define $X^k f(\mathbf{x}) = \langle \nabla X^{k-1} f(\mathbf{x}), X(\mathbf{x}) \rangle$. The transversal points on Σ with respect to the vector fields X and Y are classified as follows:

- *Crossing region*: $\Sigma^c = \{\mathbf{x} \in \Sigma, (Xf(\mathbf{x})).(Yf(\mathbf{x})) > 0\}$, formed by crossing points.

- *Sliding region*: $\Sigma^s = \{\mathbf{x} \in \Sigma, Xf(\mathbf{x}) < 0 \,\text{and}\, Yf(\mathbf{x}) > 0\}$, formed by sliding points.

- *Escaping region*: $\Sigma^e = \{\mathbf{x} \in \Sigma, Xf(\mathbf{x}) > 0 \,\text{and}\, Yf(\mathbf{x}) < 0\}$, formed by escaping points.

Filippov's convention [5] allows to define of two kinds of limit cycles for the piecewise vector fields, the so-called *sliding limit cycles* and the *crossing limit cycles*. Sliding limit cycles contain *sliding points* on the line of discontinuity and *crossing limit cycles* contain only *crossing points*. Here we only work with crossing limit cycles, or simply *limit cycles*.

3. Proof of the results

Proof of Theorem 1.1. **The continuous case.** Consider a continuous piecewise linear differential system in $\mathbb{R}^3$ separated by one plane and formed by two centers X and Y. Without loss of generality, we can suppose that the plane is $x = 0$.

The continuous hypothesis, $X(\mathbf{x}) = Y(\mathbf{x})$ at $(x, y, z) = (0, y_0, z_0)$, provides 3 polynomial equations of degree 1 in the variables y_0 and z_0, each equation corresponding to the coordinates of the involved vector fields. They write

$$(B_1 - D_1) + (A_{12} - C_{12})y_0 + (A_{13} - C_{13})z_0 = 0$$

$$(B_2 - D_2) + (A_{22} - C_{22})y_0 + (A_{23} - C_{23})z_0 = 0$$

$$(B_3 - D_3) + (A_{32} - C_{32})y_0 + (A_{33} - C_{33})z_0 = 0$$

Thus we get 9 cubic equations for the parameters, looking at each coefficient, $B_i - D_i = 0$, $A_{i2} - C_{i2} = 0$ and $A_{i3} - C_{i3} = 0$ for $i = 1, 2, 3$. From those equations we get 7 parameters b_4, b_1, a_1, b_2, a_2, α_1, β_1 in terms of the other ones, as solution of the system.

If the continuous piecewise differential system has a limit cycle, this must intersect the plane $x = 0$ in two points $(0, y_0, z_0)$ and $(0, y_1, z_1)$ with $(y_0, z_0) \neq (y_1, z_1)$. For the vector field X we have the first integrals

$$F_1(\mathbf{x}) = (a_4 + a_1x + a_2y + a_3z)^2 + (b_4 + b_1x + b_2y + b_3z)^2,$$

$$F_2(\mathbf{x}) = c_4 + c_1x + c_2y + c_3z,$$

and for the vector field Y the first integrals

$$G_1(\mathbf{x}) = (\alpha_1x + \alpha_2y + \alpha_3z + \alpha_4)^2 + (\beta_1x + \beta_2y + \beta_3z + \beta_4)^2,$$

$$G_2(\mathbf{x}) = \gamma_1x + \gamma_2y + \gamma_3z + \gamma_4.$$

Clearly that the two points $(0, y_0, z_0)$ and $(0, y_1, z_1)$ with $(y_0, z_0) \neq (y_1, z_1)$ where the limit cycles intersects the plane $x = 0$ must satisfy the system of four equations

$$e_1 = F_1(0, y_0, z_0) - F_1(0, y_1, z_1) = 0,$$

$$e_2 = F_2(0, y_0, z_0) - F_2(0, y_1, z_1) = 0,$$

$$e_3 = G_1(0, y_0, z_0) - G_1(0, y_1, z_1) = 0,$$

$$e_4 = G_2(0, y_0, z_0) - G_2(0, y_1, z_1) = 0.$$

Since the unique solution of this system is $y_0 = y_1$ and $z_0 = z_1$, it follows that the continuous piecewise differential system formed by the centers X and Y has no limit cycles.

The discontinuous case. Assume that we have a discontinuous piecewise linear differential system in $\mathbb{R}^3$ separated by one plane and formed by two centers X and Y. Without loss of generality we can suppose that the plane is $x = 0$.

Then a limit cycle intersects the plane $x = 0$ in the two points $(0, y_0, z_0)$ and $(0, y_1, z_1)$ with $(y_0, z_0) \neq (y_1, z_1)$. The vector fields X and Y have the first integrals $F_1(\mathbf{x}), F_2(\mathbf{x})$ and $G_1(\mathbf{x}), G_2(\mathbf{x})$ respectively, given in the proof of Theorem 1.1. As in that proof the two points $(0, y_0, z_0)$ and $(0, y_1, z_1)$ must satisfy

$$e_1 = F_1(0, y_0, z_0) - F_1(0, y_1, z_1) = 0,$$

$$e_2 = F_2(0, y_0, z_0) - F_2(0, y_1, z_1) = 0,$$

$$e_3 = G_1(0, y_0, z_0) - G_1(0, y_1, z_1) = 0,$$

$$e_4 = G_2(0, y_0, z_0) - G_2(0, y_1, z_1) = 0.$$

Computing the Gröebner basis of the polynomials e_1, e_2, e_3 and e_4 respect to the variables y_0, z_0, y_1 and z_1, we obtain an equivalent polynomial system to $e_1 = e_2 = e_3 = e_4 = 0$ formed by 59 equations. One of those ones is $(z_1 - z_0)(c_3\gamma_2 - c_2\gamma_3) = 0$.

Assume that $c_3\gamma_2 - c_2\gamma_3$ is not zero. Then $z_1 = z_0$. Therefore $e_4 = (y_0 - y_1)\gamma_2$. Since y_1 cannot be equal to y_0, otherwise the point $(0, y_0, z_0)$ would be equal to $(0, y_1, z_1)$, we have that $\gamma_2 = 0$. Then $e_2 = (y_0 - y_1)c_2 = 0$, so $c_2 = 0$, in contradiction with the assumption that $c_3\gamma_2 - c_2\gamma_3$ is not zero. Therefore in what follows we can assume that $c_3\gamma_2 - c_2\gamma_3 = 0$ and that $z_0 \neq z_1$. Now we consider two cases.

Case 1: c_2 is not zero. Then $\gamma_3 = c_3\gamma_2/c_2$. Doing again the Gröebner basis of the polynomials e_1, e_2, e_3 and e_4 respect to the variables y_0, z_0, y_1 and z_1, we obtain an equivalent polynomial system with eight equations. After removing the non-zero factor $z_0 - z_1$ from the equations having such a factor, all these are linear in the variables y_0, z_0, y_1 and z_1 except one equation. These equations can be solved and we obtain a continuum of solutions. Consequently, again we do not have limit cycles in this case.

Case 2: $c_2 = 0$. Then $e_2 = c_3(z_0 - z_1) = 0$. Hence $c_3 = 0$. It remains three equations $e_1 = e_3 = e_4 = 0$ and four unknowns. we obtain a continuum of solutions. Consequently again we do not have limit cycles in this case. □

Proof of Theorem 1.2. The continuous hypotheses that $X(\mathbf{x}) = Y(\mathbf{x})$ at $(x, y, z) = (1, y_0, z_0)$ and $Y(\mathbf{x}) = Z(\mathbf{x})$ at $(x, y, z) = (-1, y_1, z_1)$, provide 6 polynomial equations of degree 1 in the variables y_0, z_0, y_1 and z_1, each equation corresponding to the coordinates of the involved vector fields. Thus, similarly to the proof of Theorem 1.1, we get 18 equations for the parameters, looking at each coefficient. From that system we get 13 parameters, which satisfy the whole equations, b_4, b_1, a_1, b_2, a_2, α_1, β_1, B_4, B_1, A_1, B_2, A_2 and γ_1 in terms of the other ones.

The vector fields X and Y have the first integrals $F_1(\mathbf{x})$, $F_2(\mathbf{x})$ and $G_1(\mathbf{x})$, $G_2(\mathbf{x})$, respectively. For the vector field Z we have the firsts integrals

$$H_1(\mathbf{x}) = (A_1x + A_2y + A_3z + A_4)^2 + (B_1x + B_2y + B_3z + B_4)^2,$$

$$H_2(\mathbf{x}) = C_1x + C_2y + C_3z + C_4.$$

A possible limit cycle intersects at the points $(1, y_0, z_0)$ and $(1, y_3, z_3)$ the plane $x = 1$, and at the points $(-1, y_1, z_1)$ and $(-1, y_2, z_2)$ in the plane $x = -1$, with $(y_0, z_0) \neq (y_3, z_3)$ and $(y_1, z_1) \neq (y_2, z_2)$. These four points have to satisfy the following system of equations for (y_0, y_1, y_2, y_3, z_0, z_1, z_2, z_3).

$$e_1 = F_1(1, y_3, z_3) - F_1(1, y_0, z_0) = 0,$$

$$e_2 = F_2(1, y_3, z_3) - F_2(1, y_0, z_0) = 0,$$

$$e_3 = G_1(1, y_0, z_0) - G_1(-1, y_1, z_1) = 0,$$

$$e_4 = G_2(1, y_0, z_0) - G_2(-1, y_1, z_1) = 0,$$

$$e_5 = H_1(-1, y_1, z_1) - H_1(-1, y_2, z_2) = 0,$$

$$e_6 = H_2(-1, y_1, z_1) - H_2(-1, y_2, z_2) = 0,$$

$$e_7 = G_1(-1, y_2, z_2) - G_1(1, y_3, z_3) = 0,$$

$$e_8 = G_2(-1, y_2, z_2) - G_2(1, y_3, z_3) = 0.$$

Using the even equations we get that

$$\begin{aligned} z_1 = &\ \frac{2(c_1C_2\gamma_3 - c_1\gamma_2C_3 - C_1c_2\gamma_3 + C_1\gamma_2c_3)}{\gamma_3(C_2c_3 - c_2C_3)} + \frac{\gamma_2 y_0}{\gamma_3} - \frac{\gamma_2 y_1}{\gamma_3} + z_0, \\ z_2 = &\ \frac{2(c_1C_2\gamma_3 - c_1\gamma_2C_3 - C_1c_2\gamma_3 + C_1\gamma_2c_3)}{\gamma_3(C_2c_3 - c_2C_3)} - \frac{y_1(\gamma_2C_3 - C_2\gamma_3)}{\gamma_3C_3} \\ &- \frac{C_2y_2}{C_3} + \frac{\gamma_2y_0}{\gamma_3} + z_0, \end{aligned}$$

$$\begin{aligned} y_3 = &\ -\frac{c_3y_1(\gamma_2C_3 - C_2\gamma_3)}{C_3(\gamma_2c_3 - c_2\gamma_3)} + \frac{c_3y_2(\gamma_2C_3 - C_2\gamma_3)}{C_3(\gamma_2c_3 - c_2\gamma_3)} + y_0, \\ z_3 = &\ \frac{c_2y_1(\gamma_2C_3 - C_2\gamma_3)}{C_3(\gamma_2c_3 - c_2\gamma_3)} - \frac{c_2y_2(\gamma_2C_3 - C_2\gamma_3)}{C_3(\gamma_2c_3 - c_2\gamma_3)} + z_0. \end{aligned}$$

Note that we are assuming that all the denominators which appear in the previous expressions of z_1, z_2, y_3 and z_3 are non-zero. That is, we are solving the system $e_1 = 0, \dots, e_8 = 0$, in the more generic case, i.e. when the mentioned denominators do not vanish, and the denominators which will appear solving the remaining equations $e_1 = e_3 = e_5 = e_7 = 0$ do not vanish.

Then we have $e_1 = e_3 = e_5 = e_7 = 0$ for solving y_0, y_1, y_2, z_0. From $e_1 = 0$ and $e_3 = 0$ we can find y_2 and z_0, respectively. Substituting y_2 and z_0 in e_5 and e_7, we find that $y_1 = y_0 + 2(C_1c_3 - c_1C_3)/(C_2c_3 - c_2C_3)$ vanishes both equations. So we have a continuum of periodic orbits and no limit cycles. □

Proof of Theorem 1.3. We consider such a possible limit cycle which intersects at the points $(1, y_0, z_0)$ and $(1, y_3, z_3)$ the plane $x = 1$ and at the points

$(-1, y_1, z_1)$ and $(-1, y_2, z_2)$ the plane $x = -1$, with $(y_0, z_0) \neq (y_3, z_3)$ and $(y_1, z_1) \neq (y_2, z_2)$. The vector fields X, Y and Z have the first integrals $F_1(\mathbf{x}), F_2(\mathbf{x})$, $G_1(\mathbf{x}), G_2(\mathbf{x})$ and $H_1(\mathbf{x}), H_2(\mathbf{x})$, respectively.

These four points must satisfy the following systems of equations for the variables $(y_0,\, y_1,\, y_2,\, y_3,\, z_0,\, z_1,\, z_2,\, z_3)$.

$$e_1 = F_1(1, y_3, z_3) - F_1(1, y_0, z_0) = 0,$$

$$e_2 = F_2(1, y_3, z_3) - F_2(1, y_0, z_0) = 0,$$

$$e_3 = G_1(1, y_0, z_0) - G_1(-1, y_1, z_1) = 0,$$

$$e_4 = G_2(1, y_0, z_0) - G_2(-1, y_1, z_1) = 0,$$

$$e_5 = H_1(-1, y_1, z_1) - H_1(-1, y_2, z_2) = 0,$$

$$e_6 = H_2(-1, y_1, z_1) - H_2(-1, y_2, z_2) = 0,$$

$$e_7 = G_1(-1, y_2, z_2) - G_1(1, y_3, z_3) = 0,$$

$$e_8 = G_2(-1, y_2, z_2) - G_2(1, y_3, z_3) = 0.$$

Using the even equations we get

$$\begin{aligned}
z_1 = &\quad z_0 + \frac{2\gamma_1}{\gamma_3} + \frac{y_0\gamma_2}{\gamma_3} - \frac{y_1\gamma_2}{\gamma_3},\\
z_2 = &\quad -\frac{C_2 y_2}{C_3} + z_0 + \frac{2\gamma_1}{\gamma_3} + \frac{y_0\gamma_2}{\gamma_3} - \frac{y_1(C_3\gamma_2 - C_2\gamma_3)}{C_3\gamma_3},\\
y_3 = &\quad y_0 - \frac{c_3 y_1(C_3\gamma_2 - C_2\gamma_3)}{C_3(c_3\gamma_2 - c_2\gamma_3)} + \frac{c_3 y_2(C_3\gamma_2 - C_2\gamma_3)}{C_3(c_3\gamma_2 - c_2\gamma_3)},\\
z_3 = &\quad z_0 + \frac{c2 y_1(C_3\gamma_2 - C_2\gamma_3)}{C_3(c_3\gamma_2 - c_2\gamma_3)} - \frac{c_2 y_2(C_3\gamma_2 - C_2\gamma_3)}{C_3(c_3\gamma_2 - c_2\gamma_3)}.
\end{aligned}$$

Again note that we are assuming that all the denominators which appear in the previous expressions of z_1, z_2, y_3 and z_3, are non-zero. That is, we are solving the system $e_1 = 0, \ldots, e_8 = 0$, in the more generic case, i.e. when the mentioned denominators do not vanish, and the denominators which will appear solving the remaining equations $e_1 = e_3 = e_5 = e_7 = 0$ do not vanish.

EqVar	y_0	y_1	y_2	z_0
e_1	1	1	1	1
e_3	2	2	0	1
e_5	1	1	1	1
e_7	2	2	2	1

Table 1: Maximum exponent for each variable in the polynomials e_k for $k = 1, 3, 5, 7$.

We substitute the obtained values of y_3, z_1, z_2 and z_3 in the equations $e_k = 0$ for $k = 1, 3, 5, 7$.

It remains to solve the equations $e_1 = e_3 = e_5 = e_7 = 0$ with respect to the unknowns y_0, y_1, y_2 and z_0. We do not provide the big explicit expressions of the equations $e_1 = e_3 = e_5 = e_7 = 0$, which are easy to obtain with some algebraic manipulator as Mathematica or Mapple.

The equations $e_1 = 0$ and $e_5 = 0$ are both of degree one, see Table 1. Thus we can find the variables y_0 and y_1 as linear functions of the variables y_2 and of z_0.

Substituting the expressions of y_0 and y_1 into the remaining equations $e_3 = 0$ and $e_7 = 0$ we obtain two polynomial equations of degree two in the variables z_0 and y_2. Again we do not provide the huge explicit expressions of the equations $e_3 = 0$ and $e_7 = 0$ which will need several pages for writing them.

Applying the Bézout Theorem (see [34]) to this system we know that at most there are four real solutions for (y_2, z_0), which can produce four limit cycles substituting these solutions in the previously obtained expressions of $(y_0, y_1, y_3, z_1, z_2, z_3)$. Hence the first part of Theorem 1.3 is proved.

In order to complete the proof of Theorem 1.3 we provide a discontinuous piecewise differential system in $\mathbb{R}^3$ formed by three centers and separated by two parallel planes, obtained numerically following the ideas described in [35] with an absolute tolerance of 10^{-15}. Using the notation of the first part of the proof of Theorem 1.3, the center given by the vector field X is obtained for the values of the parameters

$$
\begin{aligned}
&a_1 = 13.708561862548212.., && b_1 = -2.0668139685572826..,\\
&a_2 = -0.46436203202470083.., && b_2 = -2.1162121013313744..,\\
&a_3 = -1.5737561143110694.., && b_3 = -0.30359559321560803..,\\
&a_4 = -5.2597752179200175.., && b_4 = -3.837509139027315..,\\
&c_1 = 1, \quad c_2 = -0.938643702906351.., && c_3 = 1, \quad c_4 = 1.
\end{aligned}
$$

For the vector field Y we have that

$$
\begin{aligned}
&\alpha_1 = 2, \quad \alpha_2 = -5, && \alpha_3 = -7, \quad \alpha_4 = -11,\\
&\beta_1 = -3, \quad \beta_2 = 1, && \beta_3 = 3, \quad \beta_4 = 1,\\
&\gamma_1 = 0.4545454545454545454, && \gamma_2 = -0.5454545454545444, \ \gamma_3 = 1, \ \gamma_4 = 6.
\end{aligned}
$$

And the vector field Z is given by

$$
\begin{aligned}
&A_1 = -3.840000000006582, && A_2 = 5, \quad A_3 = 0, \quad A_4 = 14,\\
&B_1 = 1, \quad B_2 = 12, && B_3 = 11, \quad B_4 = 14,\\
&C_1 = 2, \quad C_2 = -4, && C_3 = 5, \quad C_4 = -8.
\end{aligned}
$$

When we have computed (y_3, z_1, z_2, z_3) from $e_2 = e_4 = e_6 = e_8 = 0$ in the last proof, it remains the equations $e_1 = e_3 = e_5 = e_7 = 0$ for computing (y_0, y_1, y_2, z_0). From the equation $e_1 = 0$ the variable y_2 appears linearly and we obtain $y_2 = P_{y_2}(y_0, y_1, z_0)$. Now from the equation $e_5 = 0$ the variable y_1 appears linearly and we have $y_1 = P_{y_1}(y_0, z_0)$. From the polynomial equations $e_3 = e_7 = 0$ both of degree two, we can obtain the four solutions for (y_0, z_0), and from these four solutions we obtain the intersection points (y_0, z_0, y_1, z_1, y_2, z_2, y_3, z_3) of our limit cycles with the two planes of discontinuity, see Figure 1, which are

$$
\begin{gathered}
(0.8978733932366719.., -2.2444618685679187.., -0.8476262083331095..,\\
-2.2874616512423445.., 0.885305277605572.., -0.901116462491399..,\\
2.0197502924959307.., -1.1914191816421125..),
\end{gathered}
$$

$$
\begin{gathered}
(2.618593513041539.., -2.017041157299665.., 1.6304557768236465..,\\
-1.6469344679639704.., -0.25095559475947454.., -3.152063565230467..,\\
1.4005927017518383.., -3.1603099489515696..),
\end{gathered}
$$

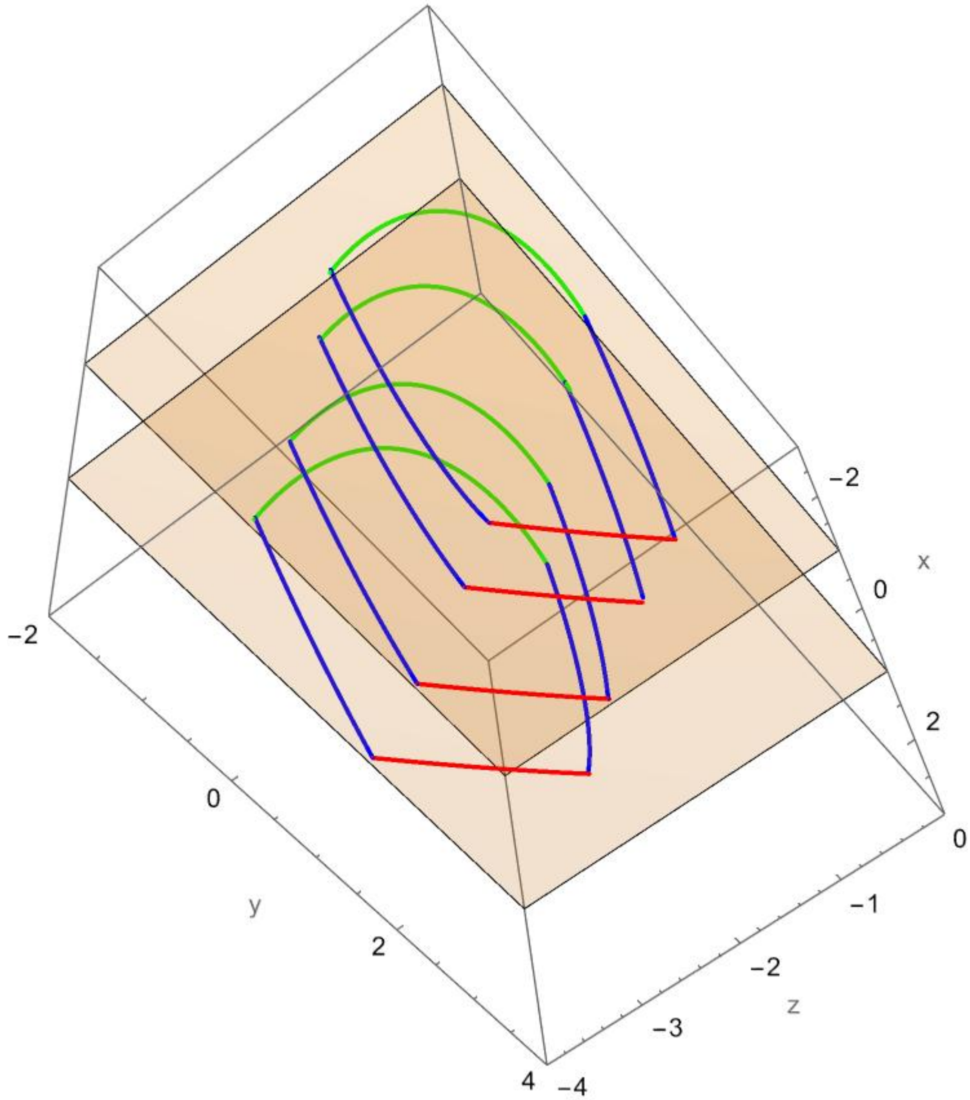

Figure 1: The four limit cycles, vector fields X (red), Y (blue) and Z (green).

$$(0.5348395678273713.., -1.7034390938102253.., -1.2941046025617466.., -1.7919540958406528.., 0.47916792990915597.., -0.3733360698639303.., 1.6828327264792675.., -0.6258825444620513..),$$

$$(1.7558413358089768.., -3.9459570275133773.., 0.15273809964405125.., -3.911286065421518.., 2.3088449504708346.., -2.1864005847600914.., 3.1516763632780256.., -2.635765268683441..).$$

□

4. Conclusions

We have shown that from the four classes of piecewise differential systems in $\mathbb{R}^3$ here studied, generically the only ones having limit cycles are the discontinuous piecewise differential systems in $\mathbb{R}^3$ separated by two parallel planes and formed by three centers having at most four limit cycles, see Theorem 1.3.

We remark that the discontinuous piecewise differential systems in $\mathbb{R}^2$ separated by two parallel straight lines and formed by three linear differential centers can have at most one limit cycle, see [28].

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