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Research paper

Hopf bifurcation in 3-dimensional polynomial vector fields

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ABSTRACT

In this work we study the local cyclicity of some polynomial vector fields in $\mathbb{R}^3$. In particular, we give a quadratic system with 11 limit cycles, a cubic system with 31 limit cycles, a quartic system with 54 limit cycles, and a quintic system with 92 limit cycles. All limit cycles are small amplitude limit cycles and bifurcate from a Hopf type equilibrium. We introduce how to find Lyapunov constants in $\mathbb{R}^3$ for considering the usual degenerate Hopf bifurcation with a parallelization approach, which enables to prove our results for 4th and 5th degrees.

1. Introduction

In 1900, D. Hilbert published a series of problems which would be very influential for mathematics during the 20th century. Ten of these problems were presented at the International Congress of Mathematicians in Paris. Among them there is the 16th Hilbert Problem, a question related to finding the maximum number of limit cycles $\mathcal{H}(n)$ that a planar polynomial system can have as a function of its degree n. See more details in the review of Y. Ilyashenko in [1]. This problem remains unsolved, but a number of researchers have made a lot of advances in this problem. J. Li did a good review of the state of the problem in [2]. About global lower bounds, the work of C. Christopher and N. Lloyd in [3] is remarkable, improved some years ago by M. Han and J. Li in [4]. Regarding summaries of known lower bounds for $\mathcal{H}(n)$ for low values of the degree, the best ones can be found in [5].

There is a local version of the Hilbert problem that consists on finding the maximum number of limit cycles of small amplitude that bifurcate from an equilibrium point for a planar polynomial vector field of degree n. This number is usually called the cyclicity of the equilibrium. The most standard way to get lower bounds for this number is to analyze the local return map defined in a neighborhood of a monodromic equilibrium point, usually by studying the maximum codimension of a degenerated Hopf bifurcation. The most recent progress in this problem for small degrees can be found in [6], studying this bifurcation near very special centers that have high codimension.

In this paper, we consider the Hopf bifurcation in families of polynomial differential systems of equations in $\mathbb{R}^3$, and we aim to find as many limit cycles as possible for systems of several degrees n. It is widely known that, unlike for planar systems, systems in $\mathbb{R}^3$ can exhibit infinitely many limit cycles, as it is the case, for example, in any vector field with a

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