NEW LOWER BOUNDS OF THE NUMBER OF CRITICAL PERIODS IN REVERSIBLE CENTERS

IVÁN SÁNCHEZ-SÁNCHEZ AND JOAN TORREGROSA

ABSTRACT. In this paper we aim to find the highest number of critical periods in a class of planar systems of polynomial differential equations for fixed degree having a center. We fix our attention to lower bounds of local criticality for low degree planar polynomial centers. The main technique is the study of perturbations of reversible holomorphic (isochronous) centers, inside the reversible centers class. More concretely, we study the Taylor developments of the period constants with respect to the perturbation parameters. First, we see that there are systems of degree $3 \leq n \leq 16$ for which up to first order at least $(n^2 + n - 4)/2$ critical periods bifurcate from the center. Second, we improve this number for centers with degree from 3 to 9. In particular, we obtain 6 and 10 critical periods for cubic and quartic degree systems, respectively.

1. INTRODUCTION

Huygens, with his work on the cycloidal pendulum in the 17th century, was the forerunner of isochronicity studies and aroused the interest of this line of research, see [2]. In the last 30 years many authors have studied the existence of differential equations with equilibrium points of center type that satisfy this isochronicity property, see for example [10, 21] and the interesting survey of Chavarriga and Sabatini [3]. There are other two very related problems, the monotonicity of the period and the bifurcation of critical periods. In this paper we deal with the second one. Before knowing with more detail such problems we need some preliminary concepts definitions and classical results in this research line.

Let us consider a real analytical system of differential equations in the plane with a center at the origin and nonzero linear part. It is a well known fact that, by a suitable change of coordinates and time rescaling, it can be written in the form

$$(\dot{x}, \dot{y}) = (-y + X(x, y), x + Y(x, y)), \tag{1}$$

where X and Y are convergent real series which start at least with quadratic monomials. We define the *period annulus* of a center as the largest neighborhood Ω of the origin with the property that the orbit of every point in $\Omega \setminus \{(0, 0)\}$ is a simple closed curve that encloses the origin, so the trajectory of every point in $\Omega \setminus \{(0, 0)\}$ is a periodic function. Suppose the origin is a center for system (1) and that the number $\rho^* > 0$ is so small that the segment $\Sigma = \{(x, y) : 0 < x < \rho^*, y = 0\}$ of the x-axis lies wholly within the period annulus. For ρ satisfying $0 < \rho < \rho^*$, let $T(\rho)$ denote the least period of the trajectory through $(x, y) = (\rho, 0) \in \Sigma$. The function $T(\rho)$ is the *period function* of the center, which by the Implicit Function Theorem is real analytic. Moreover, we say that the center of system (1) is isochronous if its period function $T(\rho)$ is constant, which means that every periodic orbit in a neighborhood of the origin has the same period.

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By performing a change to polar coordinates, one can deduce that the period function takes the form

$$T(\rho) = 2\pi\left(1 + \sum_{k=1}^{\infty} \mathcal{T}_k \rho^k\right), \tag{2}$$

where the $\mathcal{T}_k$ are known as the *period constants* of the center, see for example [25]. In the next section we will see how to compute these period constants. In the case that (1) depends on some parameters, the period constants are polynomials on them ([9]). A direct consequence of (2) is that, in the considered situation, system (1) has an isochronous center at the origin if and only if $\mathcal{T}_k = 0$ for all $k \in \mathbb{N}$. This result is also justified by Shafer and Romanovski in [25]. This shows that the period constants play the same role when studying isochronicity as Lyapunov constants when characterizing centers. In fact, every value $\rho > 0$ for which $T'(\rho) = 0$ is called a *critical period*. In addition, if it is a simple zero of T', i.e. $T''(\rho) \neq 0$, we call it a *simple* or *hyperbolic critical period*. Critical periods are actually the oscillations of the period function.

The first of the aforementioned problems, the monotonicity of the period function (2), is usually studied in polynomial center families. See for example [5, 28, 30]. About the second problem there are many works when the center family is fixed to be in a class of polynomials of low degree. The uniqueness of critical periods is studied for example in [12] for a class of polynomial complex centers. Recently, this uniqueness problem has also been considered for some Hamiltonian and quadratic Loud families in [24, 30]. For the quadratic family we recommend the nice work done by Chicone and Jacobs in [6]. The study of critical periods for classical quadratic Loud family was extended to some generalized Loud's centers, see [22]. For cubics, in particular for homogenenous cubics nonlinearities, we refer the reader to [15, 26]. For more information on the period function and the criticality problem we suggest the reading of [19] and [25].

The *problem of bifurcations of critical periods* or *criticality problem* is addressed to find the maximum number of zeros of T' which can bifurcate. We will focus on the bifurcation of *local* critical periods near the origin in the class of *time-reversible*, or simply *reversible*, planar polynomial vector fields of degree n. Without loss of generality, we can consider only differential systems which are invariant under the change $(x, y, t) \mapsto (x, -y, -t)$. This classic reversibility makes the system have a symmetry with respect to the straight line $y = 0$. Let us denote by $\mathcal{C}_\ell(n)$ the maximum number of local critical periods that can bifurcate from an n-th degree reversible system; our aim is to find the highest possible lower bounds of this number for different values of the degree n. This question is considered in analogy to the cyclicity problem, whose purpose is to find the maximum number of limit cycles –these are zeros of the Poincaré map– that bifurcate from a system. Observe that the concept of hyperbolic critical period is also defined in analogy to a hyperbolic limit cycle, following the idea of having multiplicity one.

The main objective of this paper is to find the highest possible lower bound for $\mathcal{C}_\ell(n)$. The problem of finding the maximum number of local critical periods which can bifurcate from a plane vector field is completely solved only for the quadratic case $n = 2$. This is done by Chicone and Jacobs in [6]: their result states that $\mathcal{C}_\ell(2) = 2$. To the best of our knowledge, for cubic reversible systems the highest number of critical periods achieved so far is 6, a result given in [33] by Yu and Han. In the case of Hamiltonian systems, [34] shows that such bound increases to 7. There are also a few works dealing with lower bounds for general families of degree n. One is given by Cima, Gasull, and da Silva in [8] proving that $\mathcal{C}_\ell(n) \geq 2\left[(n-2)/2\right]$, where $[\cdot]$ denotes the integer part. Another one is the bound that Gasull, Liu, and Yang propose in [13], which grows as $n^2/4$. In our work we have improved some of these bounds up to $n = 16$. Our main result is as follows.

Theorem 1. *The number of local critical periods in the family of polynomial time-reversible centers of degree n is*

$$\mathcal{C}_\ell(n) \geq \begin{cases} 6, & \text{for } n=3,\\ 10, & \text{for } n=4,\\ (n^2+n-2)/2, & \text{for } 5\leq n\leq 9,\\ (n^2+n-4)/2, & \text{for } 10\leq n\leq 16. \end{cases}$$

The essential tool for proving the above result is the local bifurcation of zeros of the first derivative of the period function (2). That is, for each degree n, finding the highest value for the multiplicity of a zero of T' and its unfolding in the corresponding reversible polynomial centers family. More concretely, by perturbing some special isochronous centers. This is as the usual mechanism for limit cycles of small amplitude in polynomial vector fields known as *degenerate Hopf bifurcation*, see also [25]. Regarding the number of parameters and using this bifurcation technique, the maximum number of critical periods we expect to find in the class of n-th degree time-reversible systems is

$$\mathcal{C}_\ell(n) = \frac{n^2+3n-6}{2}. \tag{3}$$

This is the value obtained in Theorem 1 for $n=3$, and it is only one more than our lower bound for $n=4$. In later sections, we will discuss more about this explicit value and why we expect that it will be the value for the maximum number of local critical periods. Observe that also for $n=2$ this value $\mathcal{C}_\ell(2)=2$ coincides with the one provided by [6] that we already mentioned. Finally, we notice that the reversible center family is one with a high amount of free parameters.

This work is devoted to prove Theorem 1 and has the following structure. Section 2 presents how to compute the period constants. In Section 3 we present a technique that can be used to increase the number of critical periods with respect to the bounds obtained by linear developments. Section 4 explains the choice of the family of isochronous centers that will be perturbed to obtain as many local critical periods as possible. All this is used in Section 5 to increase the number of local critical periods to 5 in the cubic case. Nevertheless, the complete proof of Theorem 1 for $n=3$ is done in Section 8, where it is shown that actually 6 critical periods can unfold in cubic reversible centers family, but perturbing from an isochronous center only having linear terms. Despite being a previous result (see [33]), we present an alternative proof for the existence of 6 critical periods in cubic reversible systems. With the same technique from previous sections, we also increase the number of local critical periods up to 10 for $n=4$ and the ones stated in Theorem 1 for $5\leq n\leq 9$, respectively in Sections 6 and 7. The last bounds of $\mathcal{C}_\ell(n)$ for $10\leq n\leq 16$ are also obtained in Section 7, studying only first order developments. We finish with a last short section, Section 9, where we discuss about these increment values. We notice that all the computations have been done using the computer algebra system Maple ([20]).

Finally, we would like to say a few words about the computational difficulties and what about going further in the degree n to improve Theorem 1. As we will see during the paper, some of the results have been obtained thanks to developing particular algorithms using parallelized computations. The main difficulty is related to the fact that there are no general classifications of reversible isochronous centers. Consequently, nobody knows the best one to be perturbed getting better results than Theorem 1, but using only first order developments. We have used holomorphic centers because they provide isochronous reversible centers for every degree n. But as they have many free parameters, the necessary computations to improve our main result would involve the explicit resolution of nonlinear systems of equations with several variables, concretely $n-1$ for families of degree n. This is actually the hardest point to go further in the degree.

2. Computation of the period constants

We start this section by presenting the classical mechanism to find period constants. This method can be found in [25]. We perform a change to polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ on system (1) to obtain

$$\begin{cases} \dot{r} = \sum_{k=1}^{\infty} \xi_k(\varphi) r^{k+1}, \\ \dot{\varphi} = 1 + \sum_{k=1}^{\infty} \zeta_k(\varphi) r^k, \end{cases} \tag{4}$$

where $\xi_k(\varphi)$ and $\zeta_k(\varphi)$ are homogeneous polynomials in $\sin\varphi$ and $\cos\varphi$ of degree $k+2$. Elimination of time in (4) yields to

$$\frac{dr}{d\varphi} = \sum_{k=2}^{\infty} R_k(\varphi) r^k, \tag{5}$$

where $R_k(\varphi)$ are 2π-periodic functions of φ and the series is convergent for all φ and for all sufficiently small r. The initial value problem for (5) with the initial condition $(r,\varphi) = (\rho, 0)$ has a unique solution

$$r = \rho + \sum_{k=2}^{\infty} u_k(\varphi)\rho^k, \tag{6}$$

which is convergent for all $0 \le \varphi \le 2\pi$ and all $\rho < r^*$, for some sufficiently small $r^* > 0$. The coefficients $u_k(\varphi)$ can be determined by simple quadratures. Substituting (6) into the second equation of (4) yields an equation of the form

$$\dot{\varphi} = \frac{d\varphi}{dt} = 1 + \sum_{k=1}^{\infty} F_k(\varphi)\rho^k.$$

Rewriting this equation as

$$dt = \frac{d\varphi}{1 + \sum_{k=1}^{\infty} F_k(\varphi)\rho^k} = \left(1 + \sum_{k=1}^{\infty} \Psi_k(\varphi)\rho^k\right) d\varphi$$

and integrating, we get

$$t - \varphi = \sum_{k=1}^{\infty} \theta_k(\varphi)\rho^k, \tag{7}$$

where $\theta_k(\varphi) = \int_0^{\varphi} \Psi_k(\psi)\, d\psi$ and the series in (7) converges for $0 \le \varphi \le 2\pi$ and sufficiently small $\rho \ge 0$. From (7) it follows that the least period of the trajectory of (1) passing through $(x,y) = (\rho, 0)$ for $\rho \ne 0$ is given by (2), which is the period function. Now we can directly see that the period constants $\mathcal{T}_k$ are given by the expression

$$\mathcal{T}_k = \frac{1}{2\pi}\theta_k(2\pi) = \frac{1}{2\pi}\int_0^{2\pi} \Psi_k(\psi)\, d\psi. \tag{8}$$

As we have mentioned above, this is the classical method to compute period constants. However, the integrals in (8) easily become too difficult to be explicitly obtained, so this technique is not useful in many cases for high degree polynomial vector fields. Here we present an equivalent approach which avoids integrals and reduces the problem to solving linear systems of equations. Our method is based on the ideas given in [1] and uses the Lie bracket and normal form theory.

We will consider a system in complex coordinates $z = x + \mathrm{i}\, y$ and $w = \overline{z} = x - \mathrm{i}\, y$ which is written as

$$\begin{cases} \dot{z} = \mathrm{i}\, z + Z(z,w) =: \mathcal{Z}(z,w), \\ \dot{w} = -\mathrm{i}\, w + \overline{Z}(z,w) =: \overline{\mathcal{Z}}(z,w), \end{cases} \tag{9}$$

where Z and $\overline{Z}$ are convergent series which start at least with quadratic terms and Z is a function that depends on X and Y. For the sake of simplicity, we will deal with

$$\dot{z} = \mathrm{i}\, z + Z(z,w) = \mathcal{Z}(z,w) \tag{10}$$

instead of (9), taking into account that the second component is the complex conjugate of the first one. By applying near the identity changes of variables, as the spirit of normal form transformations, system (10) can be simplified to

$$\dot{z} = \mathrm{i}\, z + \sum_{j=1}^{N} (\alpha_{2j+1} + \mathrm{i}\, \beta_{2j+1}) z (zw)^j + O_{2N+3},$$

where $N \in \mathbb{N}$ is arbitrary and $\alpha_{2j+1}, \beta_{2j+1} \in \mathbb{R}$. The above normal form can be expressed in polar coordinates as follows,

$$\begin{cases} \dot{r} = \sum\limits_{j=1}^{N} \alpha_{2j+1} r^{2j+1} + O_{2N+3}, \\ \dot{\varphi} = 1 + \sum\limits_{j=1}^{N} \beta_{2j+1} r^{2j} + O_{2N+2}. \end{cases} \tag{11}$$

As we are considering system (1), which has a center at the origin, the normal form of system (11) becomes

$$\begin{cases} \dot{r} = r^{2N+3} R(r,\varphi), \\ \dot{\varphi} = 1 + \beta_3 r^2 + \beta_5 r^4 + \cdots + \beta_{2N+1} r^{2N} + r^{2N} \Theta(r,\varphi), \end{cases} \tag{12}$$

for any $N \in \mathbb{N}$, where $\beta_3, \beta_5, \ldots, \beta_{2N+1} \in \mathbb{R}$.

The following theorems are proved in [1]. The first one establishes a relationship between these coefficients β_{2j+1} and the period constants defined in (2). The second one provides a condition to determine whether an equilibrium point having a pair of pure imaginary eigenvalues is of isochronous center type. From these results it becomes clear that coefficients β_{2j+1} play the same role as the period constants, in the sense that a center is isochronous if and only if $\beta_{2j+1} = 0$ for all $j \geq 1$.

Theorem 2 ([1]). *For all $m \geq 1$, the period constants defined in* (2) *satisfy*

(i) $\mathcal{T}_{2m-1} = 0$,

(ii) $\mathcal{T}_{2m} = 2\pi \sum\limits_{\substack{n_1+\cdots+n_l=2m \\ n_j \text{ even},\, l\geq 1}} (-1)^l \beta_{n_1+1} \cdots \beta_{n_l+1}.$

Before stating the isochronicity equivalence we recall the Lie bracket notion. We define the *Lie bracket* of two complex planar vector fields $\mathcal{Z}, \mathcal{U}$, corresponding to two real vector fields, as

$$[\mathcal{Z}, \mathcal{U}] = \frac{\partial \mathcal{Z}}{\partial z} \mathcal{U} + \frac{\partial \mathcal{Z}}{\partial w} \overline{\mathcal{U}} - \frac{\partial \mathcal{U}}{\partial z} \mathcal{Z} - \frac{\partial \mathcal{U}}{\partial w} \overline{\mathcal{Z}}. \tag{13}$$

This definition appears also in [12]. We notice that, as we have mentioned above, both vector fields $\mathcal{Z}$ and $\mathcal{U}$ are described only from their first components, because the second ones are obtained by conjugation. The first proof of the next geometrical equivalence was done by Sabatini in [27].

Theorem 3 ([1])**.** *Equation* (10) *has an isochronous center at the origin if and only if there exists* $\dot z = \mathcal{U}(z,w) = z + O(|z,w|^2)$ *such that* $[\mathcal{Z},\mathcal{U}] = 0$.

From Theorem 2, only the even period constants play a role, so we will define the m-th period constant as $T_m := \mathcal{T}_{2m}$.

Now we can bring all these results together to propose a constructive method to find the first N period constants of a system. We define

$$\mathcal{U} = z + \sum_{m=2}^{2N+1}\sum_{l=0}^{m} u_{l,m-l} z^l w^{m-l},\ \overline{\mathcal{U}} = w + \sum_{m=2}^{2N+1}\sum_{l=0}^{m} \overline{u}_{l,m-l} w^l z^{m-l},$$

and use it together with $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ in (10) to compute the Lie bracket $[\mathcal{Z},\mathcal{U}]$ from (13). Observing the structure of the normal form of a center (12) and considering Theorems 2 and 3, it is straightforward to see that we can also write the Lie bracket as

$$[\mathcal{Z},\mathcal{U}] = \widetilde{T}_1\, z(zw) + \widetilde{T}_2\, z(zw)^2 + \cdots + \widetilde{T}_N\, z(zw)^N + O_{2N+3}.$$

We have now two expressions for the Lie bracket of $\mathcal{Z}$ and $\mathcal{U}$, and equating the coefficients with the same degree from both expressions, we can constructively determine the coefficients $u_{l,m-l}$, $\overline{u}_{l,m-l}$, and $\widetilde{T}_m$ for $m = 1,\ldots,N$, simply by solving linear systems of equations. Then we have that the first nonvanishing period constants obtained above and the one provided by (8) differ only in a nonzero multiplicative constant. As both methods are equivalent for our purposes and as in this work we will use the later, for the sake of simplicity we will denote T_m instead of $\widetilde{T}_m$.

The algorithm presented here has been computationally implemented with Maple ([20]) and used to calculate all the necessary period constants to prove the results of this paper.

To end this section, we will prove the following result inspired by [17] which provides a useful method to compute the linear parts of the period constants by means of parallelization.

Proposition 4. *Consider a system, as in* (10)*, with a center at the origin*

$$\dot z = \mathrm{i}\, z + Z(z,w,\lambda), \tag{14}$$

where $\lambda = (\lambda_1,\ldots,\lambda_d) \in \mathbb{R}^d$ *are parameters such that for* $\lambda = 0$ *the origin is an isochronous center and* $Z \in \mathcal{C}^1(\lambda)$*. Assume that for every* $j = 1,\ldots,d$*, the* k*-th period constant of system* (14) *with* $\lambda_r = 0$ *for every* $r = 1,\ldots,d$ *such that* $r \neq j$ *takes the form*

$$T_k^{(j)} = \tau_k^{(j)}\lambda_j + O_2(\lambda_j),$$

for some coefficient $\tau_k^{(j)} \in \mathbb{R}$*, where* $O_2(\lambda_j)$ *denotes a sum of monomials of degree at least* 2 *in* λ_j*. Then the* k*-th period constant of system* (14) *takes the form*

$$T_k = \sum_{j=1}^{d} \tau_k^{(j)}\lambda_j + O_2(\lambda_1,\ldots,\lambda_d),$$

where $O_2(\lambda)$ *denotes a sum of monomials of degree at least* 2 *in the parameters.*

Proof. The proof is straightforward by using the linearity property in the first order terms of the period constants. The k-th period constant of system (14) must have the form

$$T_k = \sum_{j=1}^{d} \eta_k^{(j)}\lambda_j + O_2(\lambda_1,\ldots,\lambda_d),$$

for some coefficients $\eta_k^{(1)},\ldots,\eta_k^{(d)} \in \mathbb{R}$ and $O_2(\lambda_1,\ldots,\lambda_d)$ being a sum of monomials of degree at least 2 on the parameters. Now if for some $j = 1,\ldots,d$ we impose $\lambda_r = 0$

for every $r = 1, \ldots, d$ such that $r \neq j$, we obtain that the k-th period constant of the corresponding system has the form

$$T_k^{(j)} = \eta_k^{(j)} \lambda_j + O_2(\lambda_j),$$

which shows that $\eta_k^{(j)} = \tau_k^{(j)}$, where $\tau_k^{(j)}$ is as defined in the statement of the proposition. Repeating this process for every $j = 1, \ldots, d$, the statement is proved. □

Remark 5. *The structure outlined in Proposition 4 can be used together with parallelization to find the linear part of the period constants of a given center in a way which is more efficient, in computational terms, than directly applying the Lie bracket method. The idea is to consider each perturbative monomial instead of all of them together. One can separately use this method up to first order Taylor development to obtain the linear part of the corresponding k-th period constant, and then add all of them to find the linear part of T_k. It is relevant to observe that the computed linear parts are not obtained by calculating each complete period constant and then finding its power series expansion up to first order, but by directly computing its first order terms at each step.*

The advantage of this approach is that it is much easier in computational terms to find the first order part of the period constants of a number of systems with only one parameter than computing them for only one system with many parameters. As a matter of fact, what we are doing is to apply the same Lie bracket method to these simpler systems instead of directly to the initial one. Furthermore, this technique allows to parallelize the computation for each family, which allows to highly decrease the total execution time.

3. A result on the criticality of isochronous centers

Let us consider a family of isochronous centers with some parameters, and add a perturbation which keeps the center property. In this section we will prove a theorem which outlines how the criticality of such a family can increase under some conditions on the isochronicity parameters. The idea behind this result is inspired by [16] but better developed in [14], a recent work about cyclicity in families of centers. First we present a technical result which shows the structure of the first order terms of the period constants for a perturbed family of isochronous centers. This is essentially an extension of Proposition 4 adapted to the case where the unperturbed system is a parametric family of isochronous centers instead of a fixed one.

Proposition 6. *Let us consider a polynomial family of isochronous centers parametrized by $A \in \mathbb{R}^P$ for some $P \in \mathbb{N}$ and add a polynomial perturbation with coefficients $\lambda \in \mathbb{R}^N$ for some $N \in \mathbb{N}$ which does not break the center property.*

(i) The k-th period constant T_k of the perturbed system is a polynomial on the perturbative parameters λ whose coefficients are polynomials in A and takes the form

$$T_k = \sum_{j=1}^{N} g_k^{(j)}(A) \lambda_j + O_2(\lambda), \tag{15}$$

for some polynomials $g_k^{(j)}(A)$ in A which are the coefficients of the linear part of T_k with respect to λ.

(ii) Let us consider the $m \times m$ matrix $G_m(A)$ whose element in position (i, j) is $g_i^{(j)}(A)$ from expression (15). This is the matrix of coefficients of linear parts of the first m period constants. Then if $\det G_N(A) = 0$ and $\det G_{N-1}(A) \neq 0$ there exists a linear change of variables such that the first $N-1$ first period constants take the form

$$T_k = u_k + O_2(u_1, \ldots, u_N) \tag{16}$$

for $k = 1, \ldots, N-1$, where the linear part of T_k is u_k, $u_N := \lambda_N$ and $O_2(u_1, \ldots, u_N)$ denotes the higher order terms.

(iii) Under the same assumptions of (ii), the first $N + M$ period constants for some $M \in \mathbb{N}$ can be written as

$$T_k = \begin{cases} v_k, & \text{if } k = 1, \ldots, N-1, \\ \sum_{j=1}^{N-1} \widetilde{g}_k^{(j)}(A) v_j + f_{k-N}(A) u_N + O_2(v, u_N), & \text{if } k = N, \ldots, N+M, \end{cases} \tag{17}$$

where $v = (v_1, \ldots, v_{N-1})$ are new variables, $f_{k-N}(A)$ and $\widetilde{g}_k^{(j)}(A)$ are the corresponding coefficients of $v_1, \ldots, v_{N-1}, u_N$ which are rational functions in $A \in \mathbb{R}^P$, and $O_2(v, u_N)$ are analytical functions of order two in $v_1, \ldots, v_{N-1}, u_N$.

Proof. Recall that the period constants are polynomials in the parameters of the system. As parameters A do not break the isochronicity of the system they cannot appear isolated, so when considering the power series expansion of the period constant T_k, it is straightforward to see that its linear part must be a linear combination of the perturbative parameters λ with the coefficients being polynomials in A, and (i) follows.

To see (ii), as $\det G_{N-1}(A) \neq 0$, we can apply Cramer's rule to the system of $N-1$ equations $\sum_{j=1}^{N} g_k^{(j)}(A)\lambda_j = u_k$ or equivalently $\sum_{j=1}^{N-1} g_k^{(j)}(A)\lambda_j = u_k - g_k^{(N)}(A)\lambda_N =: u_k - g_k^{(N)}(A) u_N$ for $k = 1, \ldots, N-1$, with unknowns $\lambda_1, \ldots, \lambda_{N-1}$. Then we can explicitly find the linear change of variables that proves (16). By using this method it is clear that the coefficients which define the change of variables are rational functions in A.

Now let us consider new variables $v_1, \ldots, v_{N-1}$ to perform the following change, using (16), in $\mathbb{R}^N$:

$$v_k = T_k = u_k + O_2(u_1, \ldots, u_N), \quad \text{for } k = 1, \ldots, N-1.$$

As $u_1, \ldots, u_{N-1}$ are independent and have rank $N-1$, the Implicit Function Theorem can be applied to write $u_1, \ldots, u_{N-1}$ as functions of $v_1, \ldots, v_{N-1}, u_N$. This is

$$u_k = F_k(v_1, \ldots, v_{N-1}, u_N), \quad \text{for } k = 1, \ldots, N-1, \tag{18}$$

for some real functions F_k. Then by applying (15) from part (i) of the statement together with the change (18), the period constants take the form (17) where $\widetilde{g}_{N+d}^{(j)}(A)$ and $f_d(A)$ for $d = 0, \ldots, M$ and $j = 1, \ldots, N-1$ are the corresponding coefficients of $v_1, \ldots, v_{N-1}, u_N$ respectively, and are functions in $A \in \mathbb{R}^M$, and each $O_2(v, u_N)$ is an analytical function of order two in $v_1, \ldots, v_{N-1}, u_N$ due to the application of the Implicit Function Theorem. Then the statement follows. $\square$

Now we can present the aforementioned theorem.

Theorem 7. *Let us consider a polynomial family of isochronous centers parametrized by $A \in \mathbb{R}^P$ for some $P \in \mathbb{N}$ and a polynomial perturbation with coefficients $\lambda \in \mathbb{R}^N$ for some $N \in \mathbb{N}$ which does not break the center property. Let us denote by $G_m(A)$ the $m \times m$ matrix as defined in Proposition 6.*

(i) If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) \neq 0$, then the linear parts of the first period constants have rank N and at least $N-1$ simple critical periods can bifurcate.

(ii) If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) = 0$, $\det G_{N-1}(A^*) \neq 0$, $f_i(A^*) = 0$ for $i = 0, \ldots, M-1$, $f_M(A^*) \neq 0$ (where $f_0, \ldots, f_M$ are those defined in (17)) and the Jacobian determinant satisfies $J(A^*) := \det \mathrm{Jac}_{(f_0, \ldots, f_{M-1})}(A^*) \neq 0$, then M extra critical periods can bifurcate, which leads to a total of $N + M - 1$ critical periods.

Proof. If there exists $A^* \in \mathbb{R}^P$ such that $\det G_N(A^*) \neq 0$, we can apply the same technique as in Proposition 6.(ii) to obtain a change of variables to N new independent variables $u_1, \ldots, u_N$. By applying Weierstrass Preparation Theorem (see [32]), this implies that $N-1$ critical periods can bifurcate and the first statement follows.

Now let us prove statement (ii). First, as we are under the assumption $\det G_{N-1}(A^*) \neq 0$ for some $A^* \in \mathbb{R}^P$, we can apply Proposition 6 and write the first $N+M$ period constants as (17). If we set the problem in the manifold $\{v_1 = \cdots = v_{N-1} = 0\}$ –this means vanishing the first $N-1$ period constants–, the structure becomes

$$T_k = \begin{cases} 0, & \text{for } k = 1, \ldots, N-1, \\ u_N \left(f_{k-N}(A) + \sum_{l=1}^{\infty} f_{k-N}^{(l)}(A) u_N^l \right), & \text{for } k = N, \ldots, N+M, \end{cases}$$

for some functions $f_d^{(l)}(A)$ with $d = 0, \ldots, M$. As by assumption there exists $A^* \in \mathbb{R}^P$ such that the Jacobian determinant $J(A^*) \neq 0$, the Implicit Function Theorem guarantees that in a neighbourhood of $A = A^*$ and $u_N = 0$ the following change of variables can be performed in $T_N, \ldots, T_{N+M-1}$:

$$v_{N+k} = f_k(A) + \sum_{l=1}^{\infty} f_k^{(l)}(A) u_N^l, \text{ for } k = 0, \ldots, M-1.$$

As we suppose that $f_i(A^*) = 0$ for $i = 0, \ldots, M-1$ but $f_M(A^*) \neq 0$, we can rewrite

$$T_{N+k} = \begin{cases} u_N v_{N+k}, & \text{for } k = 0, \ldots, M-1, \\ u_N \left(f_M(A^*) + \sum_{l=1}^{\infty} f_M^{(l)}(A^*) u_N^l \right) =: u_N v_{N+M}, & \text{for } k = M. \end{cases}$$

Finally, by again the Implicit Function Theorem, as we have obtained M new independent variables we get the existence of M extra critical periods. $\square$

A natural consequence of the last result is the following corollary.

Corollary 8. *With the notation from Theorem 7, if* $\det G_N(A)$ *is not identically zero then generically at least* $N-1$ *simple critical periods bifurcate from the origin. The same conclusion is valid also when the number of parameters is bigger than or equal to* N*. Clearly, in this second case the corresponding matrix* G_m *would be a nonsquare matrix having rank* N*.*

Proof. The proof is straightforward by following the ideas in the proof of the previous theorem. If $\det G_N(A)$ is not identically zero, then as it is a polynomial we have that $\det G_N(A) \neq 0$ except for a set of zero Lebesgue measure, which implies that the rank of $G_N(A)$ is N and therefore $N-1$ critical periods unfold. $\square$

This last property is equivalent to the one for bifurcation of limit cycles from [7]. The idea of using just linear parts appeared previously in [6]. It is important to notice that in some cases the above determinant is identically zero, then the generic condition is never satisfied. This is the case for the analogous case of limit cycles bifurcation from holomorphic polynomial centers of degree 3, see [17].

4. The main reversible families

As we have previously mentioned, to get the bounds outlined in Theorem 1 we have considered n-th degree polynomial differential systems which are time-reversible with respect to straight lines. We can assume without loss of generality that the equilibrium is at the origin and that the symmetry line with respect to which the reversibility is considered is the horizontal axis. These differential systems take the form

$$\begin{cases} \dot{x} = -y + yf(x, y^2), \\ \dot{y} = x + g(x, y^2), \end{cases} \tag{19}$$

where $f(x, y^2)$ and $g(x, y^2)$ are polynomials in x and y of degrees $n-1$ and n, respectively. Clearly, system (19) is invariant under the classical reversibility change of coordinates $(x, y, t) \mapsto (x, -y, -t)$.

The next proposition shows that the condition of a system being reversible with respect to the horizontal axis in complex coordinates $z = x + \mathrm{i}\, y$ and $w = \overline{z} = x - \mathrm{i}\, y$ is that its coefficients are purely imaginary.

Proposition 9. *A system* (19)*, which is reversible with respect to the horizontal axis, takes in complex coordinates the form*

$$\dot{z} = \mathrm{i}\, z + \mathrm{i} \sum_{l+m\geq 2}^{n} c_{lm} z^l w^m, \tag{20}$$

where $c_{lm} \in \mathbb{R}$.

Proof. A change to complex coordinates shows that system (19) is written as

$$\begin{cases} \dot{z} = \mathrm{i}\, z + \sum\limits_{l+m\geq 2}^{n} b_{lm} z^l w^m, \\ \dot{w} = -\mathrm{i}\, w + \sum\limits_{l+m\geq 2}^{n} \overline{b}_{lm} w^l z^m, \end{cases} \tag{21}$$

for certain parameters $b_{lm} \in \mathbb{C}$ and their conjugate values $\overline{b}_{lm} \in \mathbb{C}$. Observe that the reversibility change $(x, y, t) \mapsto (x, -y, -t)$ takes the form $(z, w, t) = (x + \mathrm{i}\, y, x - \mathrm{i}\, y, t) \mapsto (x - \mathrm{i}\, y, x + \mathrm{i}\, y, -t) = (w, z, -t)$ in complex coordinates. Thus, when applied to (21), one obtains

$$\begin{cases} -\dot{w} = \mathrm{i}\, w + \sum\limits_{l+m\geq 2}^{n} b_{lm} w^l z^m, \\ -\dot{z} = -\mathrm{i}\, z + \sum\limits_{l+m\geq 2}^{n} \overline{b}_{lm} z^l w^m. \end{cases} \tag{22}$$

Now imposing that the system must remain invariant under this change, we have that systems (21) and (22) must be equal, so we see that $\overline{b}_{lm} = -b_{lm}$. The proof follows from this condition. The reversibility property in complex coordinates is given by the parameters being purely imaginary, this is $b_{lm} = \mathrm{i}\, c_{lm}$ with $c_{lm} \in \mathbb{R}$. Notice that in (20) there is no need to write the equation in $\dot{w}$ because it is the complex conjugate of the equation in $\dot{z}$. □

Let us consider an n-th degree polynomial system of the form

$$\begin{cases} \dot{z} = \mathrm{i}\, F(z, w), \\ \dot{w} = -\mathrm{i}\, \overline{F}(z, w), \end{cases} \tag{23}$$

having an isochronous center at the origin, where $F(z, w) = z + \widetilde{F}(z, w)$ being $\widetilde{F}(z, w)$ a sum of monomials of degree at least 2. As the $\dot{w}$ equation in (23) is the complex conjugate of the $\dot{z}$ equation, from now on we will simply write the equation in $\dot{z}$ to describe the system. We will also consider adding a reversible n-th degree polynomial perturbation as follows

$$\dot{z} = \mathrm{i}\, F(z, w) + \mathrm{i} \sum_{l+m\geq 2}^{n} r_{lm} z^l w^m, \tag{24}$$

where r_{lm} are real perturbative parameters so that the perturbation is reversible and thus the center property is kept.

A well-known fact is that holomorphic systems are isochronous (see [11]). We are interested in perturbing holomorphic isochronous centers by adding nonholomorphic perturbations, in which case equation (24) can be rewritten as

$$\dot{z} = \mathrm{i}\left(z + \sum_{j=2}^{n} A_j z^j\right) + \mathrm{i} \sum_{\substack{l+m\geq 2\\ m\geq 1}}^{n} r_{lm} z^l w^m, \tag{25}$$

for certain holomorphy parameters $A_j \in \mathbb{R}$, and $r_{lm} \in \mathbb{R}$ are perturbative parameters of the isochronous center

$$\dot{z} = \mathrm{i}\left(z + \sum_{j=2}^{n} A_j z^j\right), \tag{26}$$

which keep the center property due to being real.

In our work we have considered perturbations of the family of isochronous centers

$$\dot{z} = \mathrm{i}\, z \prod_{j=1}^{n-1} (1 - a_j z)\,, \tag{27}$$

where $n > 1$ and $a_j \in \mathbb{R} \setminus \{0\}$ are real parameters such that $a_j \neq a_i$ for every $i, j \in \{1, \ldots, n-1\}$, $i \neq j$. Observe that this family takes the form (26), so it is isochronous due to the holomorphy property. These systems also are Darboux linearizable, see [21, 25].

Our study will focus on reversible families of the form (27) being perturbed also inside the reversible polynomial class. The choice of these holomorphic systems is due to the fact that it is the easiest family that can be considered for any degree n. Moreover, as we will see, these particular systems are the most suitable for our study, in the sense that they provide quite a high number of oscillations of the period function without being too demanding computationally. Additionally, in the following section we also perturb some other cubic isochronous centers obtained from [4], where a complete classification of all reversible cubic isochronous centers is done.

The next result is a direct consequence of applying Theorem 7 to (25).

Theorem 10. *Consider the polynomial differential system of degree n defined in (25) with $n \geq 3$ and $A_2 = 1$. Let us denote by $G_m(A)$ the $m \times m$ matrix as defined in Proposition 6 and $N := (n^2 + n - 2)/2$ the number of perturbative parameters.*

(i) If there exists $A^* = (A_3^*, \ldots, A_n^*) \in \mathbb{R}^{n-2}$ such that $\det G_N(A^*) \neq 0$, then the linear parts of the first period constants have rank N and at least $N - 1$ simple critical periods bifurcate from the origin.

(ii) If there exists $A^* = (A_3^*, \ldots, A_n^*) \in \mathbb{R}^{n-2}$ such that $\det G_N(A^*) = 0$, $\det G_{N-1}(A^*) \neq 0$, $f_i(A^*) = 0$ for $i = 0, \ldots, M-1$, $f_M(A^*) \neq 0$ (where $f_0, \ldots, f_M$ are those defined in (17)) and the Jacobian determinant satisfies $J(A^*) := \det \mathrm{Jac}_{(f_0,\ldots,f_{M-1})}(A^*) \neq 0$, then M extra critical periods bifurcate from the origin, which leads to a total of $N + M - 1$ critical periods.

Experimentally, we have observed that we get more criticality when all the parameters A_j are nonvanishing. Then, after a variables rescaling and without loss of generality, we can fix $A_2 = 1$, when $A_2 \neq 0$. Section 7 uses the first statement fixing specific values for A. The second statement, choosing $M = n-2$, is used in Sections 5 and 6 for perturbations of holomorphic polynomial vector fields of degree 3 and 4, respectively. In these last cases we have achieved the maximum value for the corresponding criticality when $A_2 = 1$. This statement is also used in Section 7 but only with $M = 1$ for some small values of the degree n. Finally, in the above result we have not considered quadratic vector fields because this case was completely solved in [6].

5. Perturbing cubic isochronous systems

The first part of this section is focused on the cubic systems of the form (27), this is for $n = 3$. In the second part we study lower bounds for the criticality of some reversible isochronous centers appearing in [4]. We will see that at least 5 critical periods can unfold in the reversible cubic polynomial class. Actually, in Section 8 we will show that 6 critical periods can unfold in cubic systems, but not bifurcating from centers having nonlinear terms.

The first 4 critical periods appear by studying specific isochronous centers such that, after reversible perturbation, the rank of the linear parts of their period constants is 5. In all the studied cases this is the maximum found rank. Then, by Proposition 6, we can write the first 5 period constants in the form

$$T_k = u_k + O_2, \text{ for } k = 1, \ldots, 5,$$

where O_2 denotes the terms of degree at least 2, or directly $T_k = u_k$ for $k = 1, \ldots, 5$ if we use the Implicit Function Theorem. We have checked that the next three linear parts are a linear combination of these 5 variables. Consequently, in all the studied cases, no more critical periods can be found using only first order developments. We need to use higher order developments or pay attention to the nongeneric cases in some parameter families of isochronous centers.

5.1. **Perturbing holomorphic centers.**

In the next result we will study the critical periods bifurcation diagram of a 1-parameter cubic holomorphic system. We show how, by applying Theorem 10, we can obtain 5 critical periods when choosing the values for which the rank is not maximal. In the following subsection, these 5 critical periods will appear from higher order developments.

Proposition 11. *Let $a \in \mathbb{R}\setminus\{0\}$. Consider the 1-parameter family of cubic (holomorphic) reversible systems*

$$\dot{z} = \mathrm{i}\, z\,(1 - z)\,(1 - az)\,. \tag{28}$$

The number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is at least 5 for $a \in \{-3/2, -1, -2/3, 1/2, 2\}$ and 4 otherwise.

Proof. As we have explained in Section 4, system (28) is time-reversible holomorphic and therefore it has an isochronous center at the origin.

We can consider system (28) without losing generality with respect to the general cubic case (27), which is $\dot{z} = \mathrm{i}\, z\,(1 - a_1 z)\,(1 - a_2 z)$, with $|a_1| > |a_2|$. Both systems are equivalent after the rescaling $(z, w) \mapsto (a_1^{-1} z, a_1^{-1} w)$ and we get $a := a_1^{-1} a_2$. Thus, we can reduce our study to $a \in [-1, 1) \setminus \{0\}$. Notice that the case $a = 1$ is not included in (27) because $a_1 \neq a_2$.

As in (25), we consider the time-reversible cubic perturbation without the holomorphic monomials,

$$\begin{cases} \dot{z} = \mathrm{i}\, z\,(1 - z)\,(1 - az) + \mathrm{i}\,(r_{11} z w + r_{02} w^2 + r_{21} z^2 w + r_{12} z w^2 + r_{03} w^3), \\ \dot{w} = -\,\mathrm{i}\, w\,(1 - w)\,(1 - aw) - \mathrm{i}\,(r_{11} w z + r_{02} z^2 + r_{21} w^2 z + r_{12} w z^2 + r_{03} z^3). \end{cases}$$

When $a \in \mathbb{R} \setminus \{-1, 0, 1/2, 2\}$, the rank of the linear developments of first four period constants of this system with respect to $(r_{11}, r_{02}, r_{21}, r_{12})$ is 4. The explicit expressions of those linear developments are not shown here due to the fact that they are quite long. Then, after using the Implicit Function Theorem, the period constants take the form

$$T_k = u_k, \text{ for } k = 1, \ldots, 4.$$

Taking $u_1 = u_2 = u_3 = u_4 = 0$ and $r_{03} = u_5$, the fifth and sixth period constants take the form

$$\begin{aligned} T_5 &= \frac{5}{24}\,\frac{P(a)}{3a^2+2a+3}\,u_5 + u_5^2 \sum_{j=0}^{\infty} f_j(a)\,u_5^j, \\ T_6 &= -\frac{1}{42}\,\frac{Q(a)}{3a^2+2a+3}\,u_5 + u_5^2 \sum_{j=0}^{\infty} g_j(a)\,u_5^j, \end{aligned} \tag{29}$$

where $P(a) = a^3(a-2)(3a+2)(2a+3)(2a-1)$, $Q(a) = a^3(a-2)(2a-1)(834a^2+1735a+834)(a+1)^2$, and f_j and g_j are rational functions. Applying Theorem 10 we have 4 critical periods when $P(a) \neq 0$ and 5 when $P(a) = 0$, $P'(a) \neq 0$, and $Q(a) \neq 0$. Then, as $a \neq 0$, the statement follows except for the remaining cases $a \in \{-1, 1/2, 2\}$.

For the cases $a \in \{-1, 1/2, 2\}$ we need to add the holomorphic monomials, then the time-reversible cubic perturbation is now

$$\begin{cases} \dot{z} = \mathrm{i}\, z\,(1-z)\,(1-az) + \mathrm{i} \sum\limits_{k+l=2}^{3} r_{kl} z^k w^l, \\ \dot{w} = -\,\mathrm{i}\, w\,(1-w)\,(1-aw) - \mathrm{i} \sum\limits_{k+l=2}^{3} r_{kl} w^k z^l. \end{cases} \tag{30}$$

When computing the linear parts of the period constants we observe that they have rank 3 with respect to three of the parameters in $\{r_{20}, r_{11}, r_{02}, r_{30}, r_{21}, r_{12}, r_{03}\}$. Then, similarly to what we did above, we have $T_k = u_k$, for $k = 1, 2, 3$ and we should study the second order developments of T_4, T_5, T_6 under the condition $u_1 = u_2 = u_3 = 0$ with respect to the remaining parameters.

For $a = 2$ (and similarly for its equivalent case $a = 1/2$) we write the remaining parameters, as in a blowup procedure, as $r_{03} = u_4 v_1$, $r_{12} = u_4 v_2$, $r_{20} = u_4$, $r_{30} = 0$. Then,

$$T_k = u_4^2 F_{k-3}(v_1, v_2) + u_4^3 \sum_{j=0}^{\infty} f_{kj}(v_1, v_2)\, u_4^j, \text{ for } k = 4, 5, 6, \tag{31}$$

with

$$\begin{aligned} F_1(v_1, v_2) &= -96\, v_1 - \frac{304}{5}\, v_2 - 24\, v_1^2 - \frac{1016}{5}\, v_1 v_2 - \frac{1178}{15}\, v_2^2, \\ F_2(v_1, v_2) &= \frac{112}{3} v_2 - 350\, v_1^2 - \frac{922}{3}\, v_1 v_2 - \frac{1297}{126}\, v_2^2, \\ F_3(v_1, v_2) &= -\frac{1080}{7}\, v_1^2 + \frac{6264}{49}\, v_1 v_2 + \frac{212634}{1715}\, v_2^2. \end{aligned}$$

Next we show that the zero level curves of F_1 and F_2 have a transversal intersection point

$$(v_1^*, v_2^*) = \left(-\frac{6972965}{1901}\alpha^2 - \frac{807195}{7604}\alpha - \frac{1743}{3802}, \frac{105}{2}\alpha\right),$$

being α the unique simple real zero of $p(\alpha) = 5578372\alpha^3 + 183328\alpha^2 + 1789\alpha + 7$, where $F_3(v_1^*, v_2^*)$ is nonvanishing. This follows because $F_1(v_1^*, v_2^*) = F_2(v_1^*, v_2^*) = 0$,

$$F_3(v_1^*, v_2^*) = p_1(\alpha) = (1051652160\alpha^2 + 17223840\alpha + 120960)/1901 \neq 0,$$

$$\det \mathrm{Jac}_{(F_1,F_2)}(v_1^*, v_2^*) = p_2(\alpha) = (-103534584320\alpha^2 - 571544320\alpha + 7499520)/1901 \neq 0,$$

and the resultants $\mathrm{Res}(p, p', \alpha)$, $\mathrm{Res}(p, p_1, \alpha)$, and $\mathrm{Res}(p, p_2, \alpha)$ are all nonvanishing.

Then, after dividing (31) by u_4^2 and using again the Implicit Function Theorem at $(v_1, v_2, u_4) = (v_1^*, v_2^*, 0)$, we obtain that 5 critical periods unfold for this value of the parameter a.

The proof for the case $a = -1$, also considering the perturbation (30), follows similarly taking in $r_{02} = u_4$, $r_{11} = u_4 v_1$, $r_{20} = u_4 v_2$, $r_{30} = 0$. Now we have

$$\begin{aligned} F_1(v_1, v_2) &= -8 + \frac{192}{5} v_1 - \frac{16}{5} v_2, \\ F_2(v_1, v_2) &= \frac{1277}{56} + \frac{145}{24} v_1 - \frac{85}{8} v_1^2 + \frac{5}{8} v_1 v_2, \\ F_3(v_1, v_2) &= \frac{12}{35} - \frac{144}{7} v_1. \end{aligned}$$

Here, the zero level curves of F_1 and F_2 have two transversal intersection points, both of them written as

$$(v_1^*, v_2^*) = \frac{1}{5}(\alpha, 12\alpha - 5),$$

being α each simple real zero of $p(\alpha) = 42\alpha^2 - 301\alpha - 7662$. Additionally, $F_3(v_1^*, v_2^*) = p_1(\alpha) = 12(-12\alpha + 1)/35$ and $\det \mathrm{Jac}_{(F_1,F_2)}(v_1^*, v_2^*) = p_2(\alpha) = (-12\alpha + 43)/3$.

Finally, we would like to consider an alternative proof for the special case $a = -3/2$ (similarly for its equivalent case $a = -2/3$), which is a simple zero of P that does not vanish Q in (29). We will consider (30) and second order developments, as in the previous cases for which the generic result for every a does not apply.

Here, the linear parts of the first four period constants have rank 4. Then, by using the Implicit Function Theorem, $T_k = u_k$ for $k = 1, \ldots, 4$, and vanishing these first four we get the next two period constants which depend on the remaining parameters (u_5, u_6, u_7),

$$\begin{aligned} T_5 &= u_5 u_6 + O_3(u_5, u_6, u_7), \\ T_6 &= u_5 \Big(\frac{9}{2} - \frac{2552689}{12348} u_5 - \frac{1439}{245} u_6 - 15 u_7 \Big) + O_3(u_5, u_6, u_7), \end{aligned} \tag{32}$$

where $r_{03} = u_5, r_{20} = (16000u_5 - 4536u_6 - 6615u_7)/39690$, and $r_{30} = u_7$. To solve $T_5 = 0$ we need to know the different branches of the variety $T_5 = 0$ near the origin. The blowup mechanism can help to discover them. This is the procedure proposed by Loud in [18], where he considered it as a *singular use* of the Implicit Function Theorem. As we would like to find a branch where T_5 vanishes but T_6 does not, we will not use the tangent variety to $u_5 = 0$ because it is not clear from (32) whether T_6 vanishes on it or not. Then, assuming u_5 small but not zero and using the blowup $u_6 = u_5 v_1$ and $u_7 = u_5 v_2$, the expressions (32) write as

$$\begin{aligned} T_5 &= u_5^2 \Big(v_1 + u_5 \sum_{j=0}^{\infty} f_j(v_1, v_2)\, u_5^j \Big), \\ T_6 &= u_5 \Big(\frac{9}{2} + u_5 \sum_{j=0}^{\infty} g_j(v_1, v_2)\, u_5^j, \Big). \end{aligned}$$

Clearly, we can use the usual Implicit Function Theorem to write $T_5 = u_5^2 w_1$. Then, on the variety $w_1 = 0$ we have $T_5 = 0$ but $T_6 \neq 0$, and the unfolding of 5 critical periods is proved. $\square$

We notice that we have not considered $a = 0$ because in this case the unperturbed system is only quadratic, and up to first and second orders only one and two critical periods appear, respectively.

5.2. **Perturbing other isochronous.** This section is devoted to see the existence of other cubic reversible isochronous systems from which, after perturbation inside the cubic reversible class, also 5 critical periods bifurcate from the origin. All these systems appear in the full classification of cubic reversible isochronous systems of Chen and Romanovski, see [4]. We have not checked all of them and neither the ones in [3] because, as we have commented previously, we think that there will be no more critical periods bifurcating from the centers different from the harmonic oscillator.

In the following results the cubic reversible perturbations are considered as in (30), because first we switch them to complex coordinates and then we apply the mechanism explained in Section 2. Recall that the bifurcation mechanisms are the direct application of the limit cycles bifurcation mechanisms described in [7, 14].

Proposition 12. *Consider the cubic reversible isochronous systems*

$$\begin{cases} \dot{x} = -y + \frac{16}{3}xy, \\ \dot{y} = x - \frac{16}{3}x^2 + 4y^2 + \frac{256}{27}x^3, \end{cases} \qquad \begin{cases} \dot{x} = -y - 3x^2y, \\ \dot{y} = x + 2x^3 - 9xy^2. \end{cases}$$

The number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is at least 5.

Proof. The existence of the respective unfoldings of 5 critical periods follows as in Proposition 11, so we only describe the main differences.

For the first system, after using the Implicit Function Theorem we get $T_k = u_k$ for $k = 1, \ldots, 5$. Then, after vanishing them, the sixth writes as

$$T_6 = -\frac{2928640}{81}u_6^3 + O_4(u_6, u_7).$$

For the second system we need again the Implicit Function Theorem but a little more work is required. First, we get $T_k = u_k$ for $k = 1, \ldots, 3$. Then, after vanishing them and from the order two developments of the next three period constants, we have that there exists a curve in the parameters space such that, along it, the zero level curves of T_4 and T_5 intersect transversally and T_6 does not vanish at this point. The curve is defined by

$$\Lambda := (r_{02}(\lambda), r_{11}(\lambda), r_{20}(\lambda)) = \Big(\frac{3\alpha}{2}, 1, \frac{1288836\alpha^2 - 33437\alpha + 8492}{2(182687\alpha - 14408)}\Big)\lambda + O_2(\lambda),$$

being α the unique simple zero of the polynomial $p(\alpha) = 14865206\alpha^3 - 9450402\alpha^2 + 5998353\alpha - 494789$. On such curve, T_4 and T_5 vanish and

$$T_6(\Lambda) = \frac{4428675\, p_1(\alpha)}{98996508541251328(182687\alpha - 14408)^2}\lambda^2 + O_3(\lambda),$$

$$\det \mathrm{Jac}_{(T_4,T_5)}(r_{02}, r_{20})(\Lambda) = \frac{12695535\, p_2(\alpha)}{118921648(182687\alpha - 14408)^2}\lambda^2 + O_3(\lambda),$$

with

$$\begin{aligned} p_1(\alpha) &= 860144811862228359 0359\alpha^2 - 9039597241380812188234\alpha \\ &\quad + 7675022628311829 01877, \\ p_2(\alpha) &= 62303007298924\alpha^2 + 70835816547508\alpha - 7694925309941. \end{aligned}$$

Moreover, the resultants with respect to α of (p, p'), (p, p_1), and (p, p_2) are nonzero rational numbers. $\square$

Proposition 13. *Consider the cubic reversible isochronous systems*

$$\begin{cases} \dot{x} = -y + \frac{4}{3}xy, \\ \dot{y} = x - \frac{4}{3}x^2 + 4y^2 + \frac{16}{27}x^3, \end{cases} \qquad \begin{cases} \dot{x} = -y - \frac{14}{15}xy + \frac{16}{175}x^2y, \\ \dot{y} = x + \frac{16}{15}x^2 - \frac{46}{15}y^2 + \frac{64}{175}x^3 + \frac{48}{175}xy^2. \end{cases}$$

Up to a six order study, the number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is only 4.

Proof. The proof follows just by checking that the linear parts of the first five period constants have rank 5. Straightforward computations show that, after using the Implicit Function Theorem and vanishing them, the next two period constants vanish up to a six order study. □

Proposition 14. *Let $a \in \mathbb{R} \setminus \{0, \pm\sqrt{3}, \pm\sqrt{5}\}$. Consider the* 1*-parameter family of cubic isochronous reversible systems*

$$\begin{cases} \dot{x} = -y + 2(1-a^2)a^{-1}xy + 2x^2y - 2y^3, \\ \dot{y} = x + ax^2 + (2-a^2)a^{-1}y^2 + 4xy^2. \end{cases}$$

The number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is at least 5 *for $a \in \{\pm\sqrt{7/3}, \pm 2, \pm 3\}$ and* 4 *otherwise.*

Proof. The proof follows using Theorem 7 as the proof of Proposition 11. Here, the linear part of the first four period constants have rank 4, then there exists a change of variables such that $T_k = u_k$ for $k = 1, \ldots, 4$. The differences are only the expressions of T_5 and T_6 which are, after vanishing the first period constants,

$$\begin{aligned} T_5 &= -\frac{70(a-2)(a-3)(a+3)(a+2)(3a^2-7)a^4}{44a^8 + 90a^6 + 129a^4 + 167a^2 + 30} u_5, \\ T_6 &= \frac{4(834a^{10} - 16310a^8 + 115767a^6 - 387870a^4 + 629063a^2 - 401940)a^2}{44a^8 + 90a^6 + 129a^4 + 167a^2 + 30} u_5. \end{aligned}$$

□

In the above result we have not considered $a \in \{\pm\sqrt{3}, \pm\sqrt{5}\}$ because for these values more computations and higher order developments should be studied, and we suspect that no more than 5 oscillations of the period function will appear. Now we explain the main difficulties. Let $\mathcal{R}_\ell(a) = (R_1, \ldots, R_\ell)$ be the sequence of ranks of the linear developments of the ordered period constants for a fixed value of the parameter a, being $R_k = \operatorname{Rank}(T_1^{(1)}, \ldots, T_k^{(1)})$. Then, we have that $\mathcal{R}_{10}(\pm\sqrt{3}) = (1,2,3,3,4,4,4,4,4,4)$ and $\mathcal{R}_{10}(\pm\sqrt{5}) = (1,2,2,3,4,4,5,5,5,5)$ while for the other values, that is for $a \in \mathbb{R} \setminus \{0, \pm\sqrt{3}, \pm\sqrt{7/3}, \pm 2, \pm\sqrt{5}, \pm 3\}$, we have $\mathcal{R}_7(a) = (1,2,3,4,5,5,5)$.

6. Perturbing quartic isochronous systems

In this section we will prove that there exist quartic reversible centers for which at least 10 critical periods bifurcate by using Theorem 10. This proves the statement of Theorem 1 corresponding to $n = 4$, that is $\mathcal{C}_\ell(4) \geq 10$. Basically we will follow the same scheme as in the previous section for the holomorphic case. Assuming that the linear parts of the period constants of a quartic system have rank 9, we rewrite the 9 first period constants as

$$T_k = u_k + O_2, \text{ for } k = 1, \ldots, 9,$$

where the u_k are new variables which depend on the original perturbative parameters and O_2 denotes a sum of monomials of degree at least 2. Linear parts of higher period constants would be a linear combination of these u_k. For convenience we can also write directly, by using the Implicit Function Theorem, $T_k = u_k$ for $k = 1, \ldots, 9$.

Before the main result of this section we will formulate Poincaré–Miranda's Theorem, which will be necessary in the proof. This result could be described as a generalization of Bolzano's Theorem to higher dimensions.

Theorem 15 ([23])**.** *(Poincaré–Miranda's Theorem) Let* $\mathcal{B} = \{x \in \mathbb{R}^m : |x_j| \le L, \text{ for } 1 \le j \le m\}$ *and suppose that the mapping* $F = (f_1, f_2, \ldots, f_m) : \mathcal{B} \to \mathbb{R}^m$ *is continuous on* $\mathcal{B}$ *such that* $F(x) \neq (0, 0, \ldots, 0)$ *for* x *on the boundary* $\partial\mathcal{B}$ *of* $\mathcal{B}$, *and*

(i) $f_j(x_1, x_2, \ldots, x_{j-1}, -L, x_{j+1}, \ldots, x_m) \ge 0$ *for* $1 \le j \le m$, *and*
(ii) $f_j(x_1, x_2, \ldots, x_{j-1}, +L, x_{j+1}, \ldots, x_m) \le 0$ *for* $1 \le j \le m$.

Then, $F(x) = (0, 0, \ldots, 0)$ *has a solution in* $\mathcal{B}$.

For the proof of this theorem the reader is referred to [23] or [31]. By using it together with Theorem 10, in the following result we present a family of quartic reversible isochronous centers from which at least 10 critical periods can bifurcate.

Proposition 16. *Let* $a, b \in \mathbb{R}$. *Consider the* 2*-parameter family of quartic (holomorphic) reversible systems*

$$\dot{z} = \mathrm{i}\, z\,(1 - z)\,(1 - az)\,(1 - bz)\,. \tag{33}$$

Generically, at least 8 *critical periods bifurcate from the origin when perturbing in the class of reversible quartic centers. Moreover, in this perturbation class there exists a point* (a, b) *such that at least* 10 *critical periods bifurcate from the origin.*

Proof. System (33) is time-reversible holomorphic and therefore has an isochronous center at the origin. Let us add a time-reversible quartic perturbation with no holomorphic terms as in (25), this is, being $r_{lm} \in \mathbb{R}$,

$$\dot{z} = \mathrm{i}\, z\,(1 - z)\,(1 - az)\,(1 - bz) + \mathrm{i} \sum_{\substack{l+m \ge 2 \\ m \ge 1}}^{4} r_{lm} z^l w^m.$$

Straightforward computations show that the coefficients of the linear parts of the first 9 period constants, with respect to the only 9 perturbation parameters in the above equation, form an square matrix. Its determinant is a polynomial of degree 64 in the parameters of the family (a, b). We do not show it here because of its size. Then, the first statement follows from Theorem 10.(i).

The proof of the second statement needs more computations. After a linear change of coordinates in the parameters space we obtain that, generically, the period constants have the following form:

$$\begin{aligned}
T_k &= u_k + O_2, \text{ for } k = 1, \ldots, 8, \\
T_9 &= \frac{G(a,b)P(a,b)}{D(a,b)} u_9 + O_2, \\
T_{10} &= \frac{G(a,b)Q(a,b)}{D(a,b)} u_9 + O_2, \\
T_{11} &= \frac{G(a,b)R(a,b)}{D(a,b)} u_9 + O_2,
\end{aligned}$$

with $G(a,b) = (ab - a - b + 2)(ab - 2b^2 - a + b)(2a^2 - ab - a + b)a^3b^3$ and $P(a,b)$, $Q(a,b)$, $R(a,b)$, and $D(a,b)$ certain polynomials with rational coefficients in the variables a and b. We do not show the complete polynomials here because they are too large. They have respectively total degree $37, 39, 41$, and 37. Their number of monomials are respectively

$657, 736, 819$, and 606. Then, the second statement follows directly from Theorem 10.(ii) just checking that there exists a point (a_0, b_0) in the parameters space such that $P(a_0, b_0) = Q(a_0, b_0) = 0$, $R(a_0, b_0) \neq 0$, $\det \mathrm{Jac}_{(P,Q)}(a_0, b_0) \neq 0$, and $D(a_0, b_0) \neq 0$. To show the difficulty to find this special point, the zero level curves of the polynomials P, Q, R, and D in the square $[-1, 1]^2$ are depicted in Figure 1. The point (a_0, b_0) should be in the intersection of the red and blue curves but not in the green and black ones, although the curves are very close to see the point.

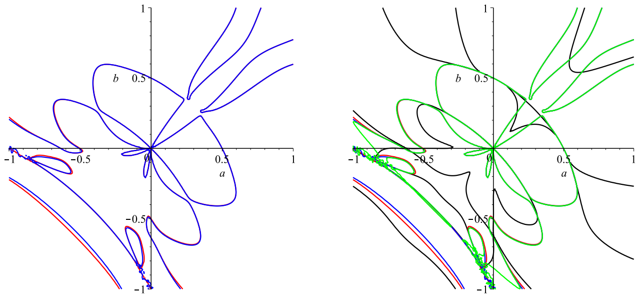

FIGURE 1. Plot of the zero level curves of $P(a, b)$, $Q(a, b)$, $R(a, b)$, and $D(a, b)$ for $(a, b) \in [-1, 1]^2$, in color red, blue, green, and black, respectively

Before proving analytically the existence of at least one intersection point (a_0, b_0), we will do some numerical simulations to later apply the Poincaré–Miranda's Theorem.

After some tedious work zooming some zones of the figure together with some tricks, we have found a numerical approximation of this special point. Increasing the number of digits in the computations up to see the stabilization of the results, we obtain

$$\begin{aligned} a_0 &\approx 0.625770358267463840706913231 27, \\ b_0 &\approx 0.71179266608573393310773491596, \\ R(a_0, b_0) &\approx -1.44391455520361722121698980760 \cdot 10^{13}, \\ \det \mathrm{Jac}_{(P,Q)}(a_0, b_0) &\approx -7.71411995359481041501433585645 \cdot 10^{29}, \\ D(a_0, b_0) &\approx -9.87896448642393578498609236141 \cdot 10^{13}. \end{aligned} \tag{34}$$

For the sake of simplicity of the expressions, we will divide each of the polynomials P, Q, R, and D by the coefficient of its highest power in a and, with a slight abuse of notation, we call them P, Q, R, and D again. Now we perform a linear change of variables which allows to separate the curves. The (numerical) Taylor expansion of $P(a, b)$ and $Q(a, b)$ at the above numerical approximation (a_0, b_0) is

$$\begin{aligned} P(a, b) \approx\ & 14476.3555282622425927110694 92 \\ & - 1516162.34376751076199474015954\, a \\ & + 1312591.63242100192712169534384\, b + O_2(a, b), \\ Q(a, b) \approx\ & 78319.0710623740477702760304 2 \\ & - 8048108.27358418867430264665612\, a \\ & + 6965439.18320811849214303073248\, b + O_2(a, b), \end{aligned}$$

where O_2 are sums of monomials of degree at least 2. Consider now a change of variables from (a, b) to new parameters (u, v) such that u and v are respectively the above linear parts. By solving these two equations with respect to a and b, we obtain that

$$\begin{aligned} a &= 0.625770358267463840706913241773 \\ &\quad + 0.00221618993488297996013588284494\, u \\ &\quad - 0.000417626554172747676923930511920\, v, \\ b &= 0.711792666085733933107734928103 \\ &\quad + 0.00256066216093940651327090078741\, u \\ &\quad - 0.000482396534881316802873914874492\, v. \end{aligned} \tag{35}$$

We notice that at $(u, v) = (0, 0)$ we approximately recover the values for (a_0, b_0) at (34). Figure 2 shows the zero level curves of the polynomials P, Q, R, and D near $(0, 0)$ after this change of variables. Now it is clear that the four zero level curves do not intersect simultaneously at such point. Moreover, the ones correspoding to P and Q are transversal. Observe that $D(u, v)$ is not seen in the graph because it stays out of the plotted region. This intersection point has shifted to near $(0, 0)$ in the new variables, and is not exactly at $(0, 0)$ due to the rounding errors.

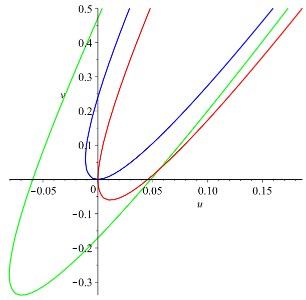

Figure 2. Plot of the zero level curves of $P(u, v)$, $Q(u, v)$, and $R(u, v)$ in color red, blue, green, respectively; the curve corresponding to $D(u, v)$ is out of the plotted region

The last step is the analytical proof of the existence of the point (a_0, b_0), which we have seen above that exists numerically. We will do a computer-assisted proof checking the properties in Theorem 15 by using rational interval analysis, because all the involved polynomials have rational coefficients. We start by writing the relation (35) as rational numbers with a 30 digits precision,

$$\begin{aligned} a &= \frac{803010141443820}{1283234545763833} + \frac{59980860399959}{27064855523371976}\, u - \frac{5287648183641}{12661187682653458}\, v, \\ b &= \frac{480154601557585}{674570874968458} + \frac{4931930765653}{1926037273048026}\, u - \frac{4470981572020}{9268270496843407}\, v. \end{aligned}$$

We will set $h = 10^{-3}$ in Theorem 15, and we will show that in the square $\mathcal{B} = [-h, h]^2$ there must be a zero of $P(u, v)$ and $Q(u, v)$. The proof follows checking also that $R(u, v)$, $D(u, v)$, and the Jacobian determinant $J(u, v) := \det \mathrm{Jac}_{(P,Q)}(u, v)$ do not vanish in the whole square. The draws in Figure 3 show that these conditions hold. Observe that $P(u, v)$ and $Q(u, v)$ are continuous because they are polynomials. Then there will be a point $(u_0, v_0) \in (-h, h)^2$ such that $P(u_0, v_0) = 0$ and $Q(u_0, v_0) = 0$ by applying the Poincaré–Miranda's Theorem because the following conditions hold.

(a) $P(h, v) > 0$ and $P(-h, v) < 0$ for $v \in [-h, h]$.
First we find the first derivatives of $P(h, v)$ and $P(-h, v)$ with respect to v. Then we compute all its real roots and see that none of them belongs to the interval $(-h, h)$, which implies that there are no local maxima nor minima in this interval. Now we

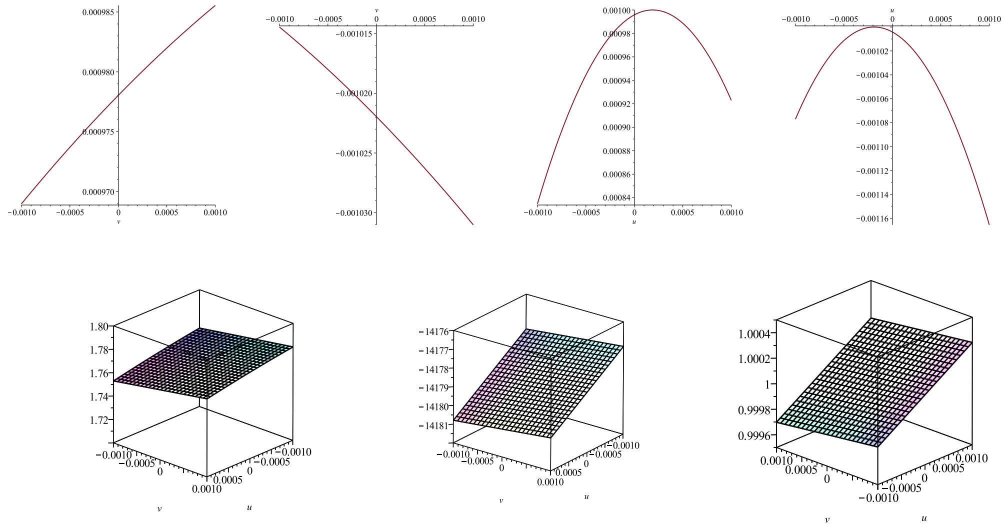

FIGURE 3. Plot of rescaled polynomials P and Q at the boundaries of $[-h,h]^2$ and the polynomials $R(u,v)$, $D(u,v)$, and $J(u,v)$ in the full square $[-h,h]^2$

check that $P(h,-h)>0$, $P(h,h)>0$, $P(-h,-h)<0$, and $P(-h,h)<0$, which together with the fact that there are not any local extrema means that the function $P(h,v)$ is strictly positive in the whole interval while $P(-h,v)$ is strictly negative.

(b) $Q(u,h)>0$ and $Q(u,-h)<0$ for $u\in[-h,h]$.
The proof follows checking that the first derivatives of $Q(u,h)$ and $Q(u,-h)$ with respect to u have only one real root in the interval $(-h,h)$, which means only one extremum. We also see that the second derivatives of $Q(u,h)$ and $Q(u,-h)$ again with respect to u at those points take a negative value, so these only local extrema are local maxima. Also, the value of $Q(u,h)$ and $Q(u,-h)$ evaluated at the u which gives the maxima are positive and negative, respectively. Additionally, $Q(-h,h)>0$, $Q(h,h)>0$, $Q(-h,-h)<0$, and $Q(h,-h)<0$. Then, the functions $Q(u,h)$ and $Q(u,-h)$ are respectively strictly positive and negative in the whole interval.

Strictly speaking, we observe that due to how Theorem 15 is formulated we should apply it to $-P(u,v)$ and $-Q(u,v)$ rather than $P(u,v)$ and $Q(u,v)$, but the conclusion is exactly the same.

The last step of the proof is to ensure that $R(u,v)$, $D(u,v)$, and $J(u,v)$ do not vanish in the whole square.

First we will prove that there exists $\widetilde{R}\in\mathbb{Q}^+$ such that $R(u,v)\geq\widetilde{R}>0$ for $(u,v)\in[-h,h]^2$. It is clear that $R(u,v)$ can be written as

$$R(u,v)=R(0,0)+\sum_{i=0}^{\hat{k}}\sum_{\substack{j=0\\(i,j)\neq(0,0)}}^{\hat{l}}a_{ij}u^iv^j \tag{36}$$

for certain rational coefficients a_{ij}, where $\hat{k}$ and $\hat{l}$ denote the degree of $R(u,v)$ with respect to u and v, respectively. Observe that

$$R(u,v)=R(0,0)+\sum_{i=0}^{\hat{k}}\sum_{\substack{j=0\\(i,j)\neq(0,0)}}^{\hat{l}}a_{ij}u^iv^j\geq R(0,0)-\sum_{i=0}^{\hat{k}}\sum_{\substack{j=0\\(i,j)\neq(0,0)}}^{\hat{l}}|a_{ij}|h^{i+j}=:\widetilde{R},$$

where we have used that $|u| \leq h$ and $|v| \leq h$. The right part of the inequality can be easily computed and we obtain a positive rational number $\widetilde{R} \approx 1.7529595059$.

The proof that there exists $\widetilde{J} \in \mathbb{Q}^+$ such that $J(u,v) \geq \widetilde{J} > 0$ for $(u,v) \in [-h,h]^2$ follows analogously to the one for $R(u,v)$, just by writing the equivalent expression (36) for function J and adequately changing the values for the degrees $\hat{k}, \hat{l}$, and the rational coefficients a_{ij}. The positive rational lower bound is $\widetilde{J} \approx 0.9996974188$. Similarly, we can prove that there exists $\widetilde{d} \in \mathbb{Q}^-$ such that $D(u,v) \leq \widetilde{d} < 0$ for $(u,v) \in [-h,h]^2$. In this case, as well as changing the values for the degrees $\hat{k}, \hat{l}$ and the rational coefficients a_{ij} we have to invert all inequalities. The upper bound is the negative rational number $\widetilde{d} \approx -14177.3096985157$.

We notice that these values for the lower and upper bounds obtained above are far from the values in (34) because we have rescaled all the involved functions. □

7. Perturbing higher degree systems

In this section we will use period constants only up to first order in the perturbative parameters to obtain as many critical periods as possible by bifurcating, in the class of reversible systems, from some reversible holomorphic systems. The idea is to consider an isochronous center of the form (27) perturbed as in (25), being $r_{lm} \in \mathbb{R}$. Using linear terms of the period constants one can deduce that at least $(n^2+n-4)/2$ critical periods bifurcate from the origin. In Proposition 17 this is proved for $3 \leq n \leq 16$. This provides the lower bound for $\mathcal{C}_\ell(n)$ given in Theorem 1 for $10 \leq n \leq 16$. In fact, we notice that for $n=3$ and $n=4$ we have already found better bounds in the previous sections, but we also include them for the sake of completeness. According to Theorem 10, under certain conditions the system could unfold up to $n-2$ extra critical periods with respect to those $(n^2+n-4)/2$ obtained by using only linear parts, as the system has $n-2$ holomorphy parameters a_j. Nevertheless, we will see that this is unfeasible even for degree 5 due to the large size of the obtained polynomials, but we will add at least one extra critical period in Proposition 18 for $5 \leq n \leq 9$. This gives the lower bound for $\mathcal{C}_\ell(n)$ given in Theorem 1 for $5 \leq n \leq 9$.

Proposition 17. *For $3 \leq n \leq 16$, consider the system*

$$\dot{z} = \mathrm{i}\, z \prod_{k=2}^{n} \left(1 - \Phi\left(\left[\frac{k}{2} \right] \right)^{(-1)^k} z \right), \tag{37}$$

where $\Phi(j)$ is the j-th prime number and $[\cdot]$ denotes the integer part function. Then, when perturbing in the class of reversible centers at least $(n^2+n-4)/2$ critical periods bifurcate from the origin, which is of isochronous reversible center type.

Proof. The n-th degree system (37) can alternatively be written as

$$\begin{cases} \dot{z} = \mathrm{i}\, z\,(1-2z)\,(1-2^{-1}z)\,(1-3z)\,(1-3^{-1}z)\,(1-5z)\,(1-5^{-1}z)\cdots, \\ \dot{w} = -\,\mathrm{i}\, w\,(1-2w)\,(1-2^{-1}w)\,(1-3w)\,(1-3^{-1}w)\,(1-5w)\,(1-5^{-1}w)\cdots. \end{cases}$$

This system is reversible and holomorphic, so it has an isochronous center at the origin. Now add an n-th degree perturbation with real parameters r_{lm} as in (25).

The next step is to compute the first $N = (n^2+n-2)/2$ period constants of the perturbed system up to first order. To this end, we apply the method presented in Section 2 which uses Proposition 4. We have performed these calculations for degree $3 \leq n \leq 16$ by using Maple plus the parallelization with PBala (see [29]), and we have found that the rank of the linear part of the first N period constants is precisely N, thus we obtain maximal rank. Therefore, by applying Theorem 10.(i) this implies that

$N-1$ critical periods bifurcate from the origin, which is the lower bound given in the statement. $\square$

It is worth saying that we would not have been able to reach degree $n=16$ in the above result without using the technique presented in Proposition 4. The reason why for a certain degree n we can obtain rank $N=(n^2+n-2)/2$ in the linear parts of the corresponding period constants is as follows. By basic combinatorics one can see that the number of perturbative terms in a reversible degree $n\geq 3$ system is

$$\sum_{j=3}^{n+1} j = \sum_{j=1}^{n+1} j - 2 - 1 = \frac{(n+2)(n+1)}{2} - 2 - 1 = \frac{n^2+3n-4}{2}. \tag{38}$$

However, observe that the terms of the form $c_{j0}z^j = A_j z^j$ belong to the holomorphic part of the system and are not considered perturbative parameters, so they cannot appear in the linear part of the period constants. As a consequence, for degree n the terms $A_j z^j$ for $2\leq j\leq n$ do not count when computing ranks of linear parts of period constants, so the number of perturbative parameters which can actually play a part results from substracting $n-1$ to the total number (38), which results in N. This means that with Proposition 17 we have reached the maximum number of critical periods that can bifurcate by studying the rank when perturbing a fixed holomorphic system using linear parts only.

As we can theoretically get rank N for degree n, then $N-1$ critical periods could bifurcate from the origin. In Proposition 17 we proved that this number of critical periods can actually appear for $3\leq n\leq 16$, and for higher degrees the problem gets too demanding in computational terms. We have used a cluster of servers with more than 100 cores and more than 300 GB of RAM in total. Nevertheless, it is natural to think that this lower bound will hold for any degree $n\geq 3$.

In the next result we provide one more critical period than the obtained in the previous proposition, considering the holomorphic reversible system of degree n

$$\dot{z} = \mathrm{i}\, z(1-z)\prod_{j=1}^{n-2}(1-a_j z)\,, \tag{39}$$

with $5\leq n\leq 9$ and $a_j\in\mathbb{R}$, but with only one free parameter instead of $n-2$, $(a_1,\ldots,a_{n-2})$, because of the difficulties in the analytical computations.

Proposition 18. *Let $5\leq n\leq 9$ be a natural number and $a\in\mathbb{R}$. For the (holomorphic) reversible 1-parameter family*

$$\dot{z} = \mathrm{i}\, z(1-az)\prod_{k=1}^{n-2}(1-kz)\,, \tag{40}$$

there exists a real value a such that at least $(n^2+n-2)/2$ critical periods bifurcate from the origin when perturbing in the class of polynomial reversible centers of degree n.

Proof. System (40) is time-reversible holomorphic and therefore it has an isochronous center at the origin. We consider the time-reversible polynomial perturbation of degree n with no holomorphic terms as in (25) and we compute the first order developments of its $(n^2+n)/2$ first period constants as a function of a. Notice that this system has $N:=(n^2+n-2)/2$ perturbative parameters, which is the maximal rank that the linear parts can have. In the case that we have rank $N-1$ instead, as in Theorem 10.(ii), a perturbative parameter is still not used. We have checked that, after a linear change of parameters, for each degree $5 \leq n \leq 9$, the period constants have the form

$$T_k = u_k + O_2 \text{ for } k = 1, \ldots, N-1,$$
$$T_N = a^2 C_n(a) \frac{P_n(a)}{D_n(a)} u_N + O_2,$$
$$T_{N+1} = a^2 C_n(a) \frac{Q_n(a)}{D_n(a)} u_N + O_2,$$

for certain polynomials $P_n(a)$, $Q_n(a)$, $D_n(a)$, and $C_n(a)$ in the variable a with rational coefficients. These polynomials are not shown here because of their large size: $P_n(a)$ has degree 100, 206, 374, 626, and 986 for $n = 5, 6, 7, 8,$ and 9, respectively; $Q_n(a)$ has degree 102, 208, 376, 628, and 988 for $n = 5, 6, 7, 8,$ and 9, respectively; $D_n(a)$ has degree 89, 188, 349, 593, and 944 for $n = 5, 6, 7, 8,$ and 9, respectively. The polynomials $C_n(a)$ are $C_5(a) = 2a - 3$ and $C_n(a) = 1$ for $n = 6, 7, 8, 9$.

To prove the unfolding of an extra critical period by following the ideas in Theorem 10, we should see that there exists some value a_n such that $P_n(a_n) = 0$, $P_n'(a_n) \neq 0$, $Q_n(a_n) \neq 0$, and $D_n(a_n) \neq 0$ for $5 \leq n \leq 9$. Straightforward computations show that $P_5(a)$ has a root a_5 in the interval $[0.75, 0.76]$, $P_6(a)$ has a root a_6 in the interval $[1.27, 1.28]$, $P_7(a)$ has a root a_7 in the interval $[0.11, 0.12]$, $P_8(a)$ has a root a_8 in the interval $[0.58, 0.59]$ and $P_9(a)$ has a root a_9 in the interval $[0.12, 0.13]$. Thus, we know that for each $n = 5, 6, 7, 8,$ and 9, $P_n(a)$ has a real root a_n.

Finally, we find that the resultant of $P_n(a)$ with $P_n'(a)$, the resultant of $P_n(a)$ with $Q_n(a)$, and the resultant of $P_n(a)$ with $D_n(a)$ are nonzero rational numbers for each $n = 5, 6, 7, 8,$ and 9, which means that $P_n(a)$ has no common zeros with $P_n'(a)$, $Q_n(a)$, and $D_n(a)$. Therefore, by applying Theorem 10.(ii), we can conclude that for degrees from 5 to 9 one extra critical period unfolds. $\square$

As we already commented, due to the fact that system (39) has $n-2$ holomorphic free parameters, according to Theorem 10.(ii) one could expect to see $n-2$ extra critical periods. However, when computing the period constants of (39) even for $n = 5$, we observe that we cannot deal with them: after appropriately handling the following three period constants T_{14}, T_{15}, T_{16}, the obtained polynomials that we need to apply the theorem have approximately half a million of monomials, with degrees $154, 156,$ and 158. Moreover, their coefficients are integer numbers between 40 and 80 digits long. Because of this, we have not been able to see numerically the existence of a transversal intersection of them. Nevertheless, by working with two parameters we have numerical evidence that for $n = 5$ actually 2 additional critical periods unfold, as we will see followingly. Even for this case, the size of the expressions is too large to achieve an analytical proof.

Let us consider the system

$$\begin{cases} \dot{z} = \mathrm{i}\, z(1-z)^2(1-az)(1-bz), \\ \dot{w} = -\,\mathrm{i}\, w(1-w)^2(1-aw)(1-bw). \end{cases} \tag{41}$$

Firstly, we compute the first 16 period constants of system (41), consider their linear parts and denote by $P(a,b)$, $Q(a,b)$, and $R(a,b)$ the numerator of the coefficient of the corresponding linear part and $D(a,b)$ the common denominator, following the notation in the proof of Proposition 18. Working with enough precision up to see the stabilization of the values of the intersection of the zero level curves of P and Q, together with the value of R, D, and the Jacobian determinant of (P,Q), we can find a transversal intersection at

$$(a_0, b_0) \approx (0.63824202454687891, -1.75185147414301379).$$

In Figure 4 we represent graphically the intersection of the zero level curves of the polynomials $P(a,b)$ and $Q(a,b)$.

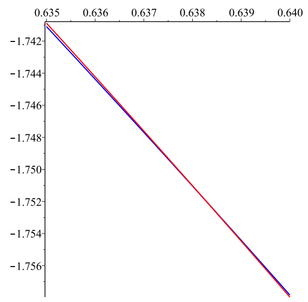

FIGURE 4. Plot of the zero level curves of polynomials $P(a,b)$ and $Q(a,b)$ in red and blue color, respectively; the zero level curves of the polynomials R, D, and the Jacobian determinant of P and Q do not appear because they stay out of figure. The intersection of $P(a,b)$ and $Q(a,b)$ can be clearly seen

8. Six critical periods on cubic systems

This section is devoted to prove the part of the statement of Theorem 1 corresponding to $n=3$, this is $\mathcal{C}_\ell(3)\geq 6$. This result will not arise from a perturbation of isochronous centers as we did in Section 5, in the sense that the perturbative parameters are not 'small'.

Proposition 19. *There exist values of* $r_{20}, r_{11}, r_{02}, r_{30}, r_{21}, r_{12}, r_{03}\in\mathbb{R}$ *for which the cubic reversible system*

$$\dot{z} = \mathrm{i}\Big(z - z^3 + \sum_{l+m=2}^{3} r_{lm} z^l w^m\Big), \tag{42}$$

unfolds 6 *local critical periods.*

Proof. The proof will consist on the following steps. First we compute the first 7 period constants of system (42). Then we show the existence of a point in the parameters space, with $r_{20}=1$, for which $T_1=\cdots=T_6=0$ but $T_7\neq 0$. The complete unfolding is proved checking that the determinant of the Jacobian matrix of $(T_1,\ldots,T_6)$ with respect to the remaining 6 parameters is not zero.

The first 7 period constants of system (42) have been obtained by using the method described in Section 2. Because of their size, here we only show the first two,

$$\begin{aligned}
T_1 ={}& -2r_{11}r_{20} + 2r_{21} - \frac{4}{3}r_{02}^2 - 2r_{11}^2,\\
T_2 ={}& 4r_{12} - 8r_{11}^2 + 4r_{11}r_{20} - 4r_{12}^2 - 3r_{03}^2 - 4r_{12}r_{30} + 8r_{11}^2r_{30} + 8r_{11}^2r_{21} + \frac{8}{3}r_{02}^2r_{21}\\
&+ 16r_{11}^2r_{12} - \frac{8}{3}r_{20}r_{02} - \frac{40}{3}r_{02}r_{11}^3 - \frac{44}{3}r_{11}r_{02} - \frac{4}{3}r_{02}^2r_{20}^2 - 15r_{02}^2r_{11}^2 + 20r_{12}r_{11}r_{02}\\
&+ \frac{8}{3}r_{02}r_{12}r_{20} + 4r_{11}r_{12}r_{20} - 4r_{11}r_{20}r_{21} + \frac{4}{3}r_{03}r_{20}r_{02} + \frac{44}{3}r_{30}r_{11}r_{02} + \frac{58}{3}r_{02}r_{03}r_{11}\\
&+ \frac{8}{3}r_{30}r_{20}r_{02} - 4r_{11}r_{20}r_{30} - \frac{28}{3}r_{02}^2r_{11}r_{20} - \frac{8}{3}r_{02}r_{20}^2r_{11} - 12r_{02}r_{20}r_{11}^2.
\end{aligned}$$

The number of monomials of the following constants, T_3, T_4, T_5, T_6, T_7, are respectively $164, 576, 1645, 3861, 8303$, and their degrees are $6, 8, 10, 12, 14$.

Now the second step is to check that there exists some point in the parameters space such that the first 6 period constants vanish but T_7 does not. Let us start by imposing $r_{20}=1$ and solving $T_1=T_2=0$ provided that $D:=3r_{12}+3r_{11}-11r_{02}r_{11}-2r_{02}-6r_{11}^2\neq 0$. Then

$$
\begin{aligned}
r_{21} =& r_{11}+\frac{2}{3}r_{02}^2+r_{11}^2,\\
r_{30} =& \frac{1}{12\left(3r_{12}+3r_{11}-11r_{02}r_{11}-2r_{02}-6r_{11}^2\right)}\left(16r_{02}^4-63r_{02}^2r_{11}^2-84r_{02}^2r_{11}\right.\\
&-12r_{02}^2-120r_{02}r_{11}^3-108r_{02}r_{11}^2-24r_{02}r_{11}^2+72r_{11}^4+36r_{11}^3-36r_{11}^2\\
&+174r_{02}r_{03}r_{11}+12r_{02}r_{03}+180r_{02}r_{11}r_{12}+24r_{02}r_{12}+144r_{11}^2r_{12}\\
&\left.+36r_{11}r_{12}-132r_{02}r_{11}-24r_{02}-27r_{03}^2-72r_{11}^2+36r_{11}-36r_{12}^2+36r_{12}\right).
\end{aligned}
$$

Under the above condition $D\neq 0$, the Jacobian determinant of T_1 and T_2 with respect to r_{21} and r_{30} is nonzero. This implies that the study of the complete versal unfolding of the 6 critical periods can be restricted to the study of the remaining period constants with respect to the four free parameters $r_{11}, r_{02}, r_{12}, r_{03}$.

To simplify the manipulation of $T_3,\dots,T_7$, we take their numerators and divide them by their highest coefficient in absolute value; with a slight abuse of notation, we call them again $T_3,\dots,T_7$, respectively.

Before the analytical proof, we will provide numerical evidence that there exists a solution for $\{T_3=0, T_4=0, T_5=0, T_6=0\}$ such that T_7, the denominator D, and the Jacobian determinant J of (T_3,T_4,T_5,T_6) do not vanish. We have increased the precision up to see the stabilization of the results. A 30-digits approximation to this intersection point is

$$
\begin{aligned}
S := \{r_{11} &= 0.332239671964981276819848124224,\\
r_{02} &= -1.14623564863006725151534814297,\\
r_{12} &= 0.707146879073682873590033571024,\\
r_{03} &= -0.857479316438844353902485565632\}
\end{aligned}
$$

and, at this point,

$$
\begin{aligned}
T_7 =& -1.84620573446485590097286118\cdot 10^{-9},\\
D =& -4.92423261813104720132211463191\cdot 10^{-14},\\
J =& -8.93740626746136868462260172503.
\end{aligned}
$$

Even though T_7 and D might seem too close to zero, the numerical values of T_3, T_4, T_5, T_6 at S are about 20 orders of magnitude lower, so we can actually consider that T_7 and D are nonzero.

Having this numerical evidence, we will proceed with the analytical proof by following a computer-assisted proof as we have done in the proof of Proposition 16.

Let us consider the rational approximation of the first order Taylor expansion of the period constants T_3, T_4, T_5, T_6 at the point S,

$$
\begin{aligned}
T_3^{(1)} &= -\frac{73352896192857}{1157958866091236} + \frac{66262571735671}{670216015479518} r_{11} - \frac{119234362424303}{776335803127460} r_{02} \\
&\quad + \frac{7903848963503}{675876388619388} r_{12} + \frac{55731331328881}{310685226195660} r_{03}, \\
T_4^{(1)} &= -\frac{25841873263308}{2144739207215017} + \frac{40160593855699}{1426912747264762} r_{11} - \frac{117691544210802}{4702223212288759} r_{02} \\
&\quad + \frac{7773205101075}{2079218586073918} r_{12} + \frac{110363479645312}{3304887976984249} r_{03}, \\
T_5^{(1)} &= -\frac{16219703349568}{10414082088666585} + \frac{42608385876433}{9737281715798994} r_{11} - \frac{29366717293918}{9762325804542787} r_{02} \\
&\quad + \frac{6748894626740}{9711312046719413} r_{12} + \frac{6430413960561}{1437481484699156} r_{03}, \\
T_6^{(1)} &= -\frac{1959207228925}{14543712037193487} + \frac{11134499629945}{27672511586934129} r_{11} - \frac{5248411486748}{20480381071923191} r_{02} \\
&\quad + \frac{3134463695044}{45208235327076605} r_{12} + \frac{8433554097025}{21161093903316966} r_{03},
\end{aligned}
$$

and the change of variables

$$
\{T_3^{(1)} = u_1, T_4^{(1)} = u_2, T_5^{(1)} = u_3, T_6^{(1)} = u_4\}. \tag{43}
$$

Now one can solve this system to obtain the inverse change. To deal with shorter rational numbers, we convert the coefficients of the resulting expressions to a 30-digit approximation and then reconvert it to rational, obtaining

$$
\begin{aligned}
r_{11} &= \frac{114216314885635}{343776871106692} + \frac{3690270297600200}{19535367373429} u_1 - \frac{24306493749268230}{4889036398111} u_2 \\
&\quad + \frac{3565552496655516}{44921198443} u_3 - \frac{745800490350068047}{133328816009} u_4, \\
r_{02} &= -\frac{2830661790614852}{2469528664544625} - \frac{8579444837165377}{6135160885376} u_1 + \frac{62633434081451044}{1884512347855} u_2 \\
&\quad - \frac{218159188810346297}{437320083677} u_3 + \frac{433555893724556147}{125890356506} u_4, \\
r_{12} &= \frac{308981620175863}{436941220161516} + \frac{14794405087051724}{7814232065819} u_1 - \frac{52171010172694907}{1180772594709} u_2 \\
&\quad + \frac{207814747586335205}{323119349554} u_3 - \frac{91326167194000251}{20903373436} u_4, \\
r_{03} &= -\frac{301994834117308}{352189059639955} - \frac{28146991231557103}{19831964964997} u_1 + \frac{65898129221474685}{1933801767914} u_2 \\
&\quad - \frac{292164620132414823}{569752028783} u_3 + \frac{570903623821683593}{161190849884} u_4.
\end{aligned}
$$

Using these expressions we can rewrite the whole $T_3, \ldots, T_7$ in these new variables. For simplicity we denote them by $U_j(u_1, u_2, u_3, u_4) := T_{j+2}(r_{11}, r_{02}, r_{12}, r_{03})$ for $j = 1, \ldots, 5$. Observe that the first order Taylor expansion of $U = (U_1, U_2, U_3, U_4)$ with respect to the variables $u = (u_1, u_2, u_3, u_4)$ is near the identity. Consequently, the problem reduces to proving the existence of some point $u^* = (u_1^*, u_2^*, u_3^*, u_4^*)$ near the origin for which $U_1(u^*) = U_2(u^*) = U_3(u^*) = U_4(u^*) = 0$, and $U_5(u^*)$, the denominator $D(u^*)$, and the Jacobian determinant $J(u^*) := \det \mathrm{Jac}_U(u^*)$ do not vanish. The existence of such point will be shown applying again Poincaré–Miranda's Theorem (Theorem 15).

Let us set $h = 10^{-12}$. We have implemented an algorithm which provides rational upper and lower bounds to a given function with m variables in $\mathcal{B} = [-h,h]^m$, for $m = 3,4$. Using it as a computer-assisted proof, we have been able to find the following bounds.

- For U_1, we have $0 < \hat{u}_1 < U_1(h,u_2,u_3,u_4)$ and $U_1(-h,u_2,u_3,u_4) < -\hat{u}_1 < 0$ for all $u_2,u_3,u_4 \in [-h,h]$, where $\hat{u}_1 \approx 2.67 \cdot 10^{-13}$.
- For U_2, we have $0 < \hat{u}_2 < U_2(u_1,h,u_3,u_4)$ and $U_2(u_1,-h,u_3,u_4) < -\hat{u}_2 < 0$ for all $u_1,u_3,u_4 \in [-h,h]$, where $\hat{u}_2 \approx 8.78 \cdot 10^{-13}$.
- For U_3, we have $0 < \hat{u}_3 < U_3(u_1,u_2,h,u_4)$ and $U_3(u_1,u_2,-h,u_4) < -\hat{u}_3 < 0$ for all $u_1,u_2,u_4 \in [-h,h]$, where $\hat{u}_3 \approx 9.85 \cdot 10^{-13}$.
- For U_4, we have $0 < \hat{u}_4 < U_4(u_1,u_2,u_3,h)$ and $U_4(u_1,u_2,u_3,-h) < -\hat{u}_4 < 0$ for all $u_1,u_2,u_3 \in [-h,h]$, where $\hat{u}_4 \approx 9.98 \cdot 10^{-13}$.

This means that U_j is positive in $u_j = h$ and negative in $u_j = -h$ for $j = 1,2,3,4$. Therefore, by applying Poincaré–Miranda's Theorem we can conclude that there exists some point in $[-h,h]^4$ which vanishes U_1,U_2,U_3,U_4.

By following an analogous computer-assisted proof, one can see that functions U_5 and D satisfy $U_5(u_1,u_2,u_3,u_4) < -\hat{u}_5 < 0$ and $D(u_1,u_2,u_3,u_4) < -\hat{d} < 0$ for all $u_1,u_2,u_3,u_4 \in [-h,h]$, where $\hat{u}_5 \approx 1.84 \cdot 10^{-9}$ and $\hat{d} \approx 8.93$, so both functions are always negative in $[-h,h]^4$ and do not vanish in the box.

The last part of the proof will be to check that the Jacobian determinant $J(u)$ is also nonzero in $[-h,h]^4$. From the change (43), it is clear that the Jacobian matrix Jac_U is close to the identity matrix I and we can write $\mathrm{Jac}_U = I + M$ for some matrix M. By adapting and using the previously implemented algorithm, we find upper and lower bounds for each one of the 16 entries (k,l) of M, proving that for every entry M_{kl} of the matrix there exists a positive rational number $\hat{m}_{kl}$ such that $-\hat{m}_{kl} < M_{kl} < \hat{m}_{kl}$.

It is straighforward to check that the Jacobian determinant $J(u)$ has the following structure,

$$J(u) = 1 + \sum_{s=1}^{64} \mathcal{M}_s,$$

where every $\mathcal{M}_s$ is a product of entries of matrix M which may be either positive or negative. Let us denote by $\hat{\mathcal{M}}_s$ the rational number resulting of the substitution of every factor M_{kl} by $\hat{m}_{kl}$ in $\mathcal{M}_s$. We have then a rational lower bound $\hat{J}$ for which $J(u)$ satisfies

$$J(u) = 1 + \sum_{s=1}^{64} \mathcal{M}_s \geq 1 - \sum_{s=1}^{64} |\hat{\mathcal{M}}_s| = \hat{J} \approx 0.9918518555136.$$

This justifies that the determinant is positive for every $u_1,u_2,u_3,u_4 \in [-h,h]$, so we can guarantee that it does not vanish in $[-h,h]^4$ and the result follows. □

9. Final remarks for arbitrary degree

The method used in Section 5 for cubics and Section 6 for quartics can be theoretically extended to systems of any degree n. We have seen that for the cubic case we can obtain families with an extra parameter which gives one extra oscillation, and for the quartic case we have families with two extra parameters which give two extra oscillations. Indeed, holomorphic reversible systems of degree n of the form (27) can be rescaled as $z \mapsto a_1^{-1} z$ to obtain

$$\dot{z} = \mathrm{i}\, z\,(1-z)\,(1-b_1 z)\cdots(1-b_{n-2}z)\,,$$

where we have defined the $n-2$ new parameters $b_j := a_{j+1}a_1^{-1}$ for $j = 1,\ldots,n-2$. By adding a time-reversible perturbation, with the same technique from Sections 5 and 6 we should be able to obtain $n-2$ extra critical periods. Even though this method seems pretty clear from a theoretical point of view, when trying to make the calculations one realises that it soon becomes too demanding in computational terms, and this is the reason why we have not gone further than $n = 4$. However, we think that these $n-2$ extra critical periods must appear near the holomorphic reversible centers, by bifurcation in the class of polynomial reversible systems of degree n. Then the *local criticality of polynomial holomorphic reversible* systems of degree n in the class of polynomial reversible vector fields also of degree n would be $\mathcal{C}_\ell^h(n) \geq (n^2+3n-8)/2$. We notice that we have not considered here the harmonic oscillator, $\dot{z} = \mathrm{i}\, z$, because it is not strictly a degree n system.

As we have seen in Section 8, if we consider the complete polynomial reversible center family of degree 3, an extra oscillation can be found when using the total number of parameters except the scaled one. This rescaling is saying that the harmonic oscillator will be the reversible center with the highest criticality. We think that what is happening for degree 3 is a bifurcation phenomenon that will occur for every degree, being $\mathcal{C}_\ell(n) \geq (n^2+3n-6)/2$.

As a summary, being $N = (n^2+3n-4)/2$ the total number of parameters in reversible nondegenerate centers, we think that $\mathcal{C}_\ell(n) \geq N-1$ while $\mathcal{C}_\ell^h(n) \geq N-2$.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona (Spain)
Email address: isanchez@mat.uab.cat

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona (Spain); Centre de Recerca Matemàtica, Campus de Bellaterra, 08193 Bellaterra, Barcelona (Spain)
Email address: torre@mat.uab.cat