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Integrability and Limit Cycles via First Integrals

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Abstract: In many problems appearing in applied mathematics in the nonlinear ordinary differential systems, as in physics, chemist, economics, etc., if we have a differential system on a manifold of dimension, two of them having a first integral, then its phase portrait is completely determined. While the existence of first integrals for differential systems on manifolds of a dimension higher than two allows to reduce the dimension of the space in as many dimensions as independent first integrals we have. Hence, to know first integrals is important, but the following question appears: *Given a differential system, how to know if it has a first integral*? The symmetries of many differential systems force the existence of first integrals. This paper has two main objectives. First, we study how to compute first integrals for polynomial differential systems using the so-called Darboux theory of integrability. Furthermore, second, we show how to use the existence of first integrals for finding limit cycles in piecewise differential systems.

Keywords: limit cycles; Darboux theory of integrability; first integrals

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1. Introduction to the Darboux Theory of Integrability

For a differential system on a two dimensional manifold, its phase portrait is determined by the existence of a first integral. The Hamiltonian differential systems are the easiest differential systems having a first integral.

A differential system of the form:

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x},$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function, is a *Hamiltonian differential system* or a simple *Hamiltonian system* in $\mathbb{R}^2$.

The integrable planar differential systems different from the Hamiltonian ones, in general, are not easy to find. First, we stated the basic results of the Darbouxian theory of integrability for finding first integrals for planar polynomial differential systems. The Darbouxian theory of integrability connects the integrability of polynomial differential systems with the invariant algebraic curves that those systems have.

1.1. Polynomial Differential Systems

Let P and Q be real polynomials in the real variables x and y. Then, a differential system:

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

is a two dimensional *planar polynomial differential system* or simply a *polynomial system*. As usual, $m = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial system. In this paper, we supposed that the polynomials P and Q were coprime in the ring of real polynomials in the variables x and y.