THE MARKUS–YAMABE CONJECTURE FOR CONTINUOUS AND DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS

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ABSTRACT. In 1960 Markus and Yamabe made the following conjecture: If a C^1 differential system $\dot{\mathbf{x}} = F(\mathbf{x})$ in $\mathbb{R}^n$ has a unique equilibrium point and the Jacobian matrix of $F(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ has all its eigenvalues with negative real part, then the equilibrium point is a global attractor. Until 1997 we do not have the complete answer to this conjecture. It is true in $\mathbb{R}^2$, but it is false in $\mathbb{R}^n$ for all $n > 2$.

Here we extend the conjecture of Markus and Yamabe to continuous and discontinuous piecewise linear differential systems in $\mathbb{R}^n$ separated by a hyperplane, and we prove that for the continuous piecewise linear differential systems it is true in $\mathbb{R}^2$, but it is false in $\mathbb{R}^n$ for all $n > 2$. But for discontinuous piecewise linear differential systems it is false in $\mathbb{R}^n$ for all $n \geq 2$.

1. Introduction and statement of the main results

For the n–dimensional differential system

$$\dot{\mathbf{x}} = A\mathbf{x} + \varphi(c^T\mathbf{x})b, \tag{1}$$

with $\varphi \in C^0(\mathbb{R})$ satisfying $\varphi(0) = 0$ and $b, c \in \mathbb{R}^n$, we say that $S(\alpha, \beta)$ is a sector of linear stability (also called *Hurwitz sector*) if for all $k \in (\alpha, \beta)$ the matrix $A + kbc^T$ is *Hurwitz* (that is all its eigenvalues have negative real parts), and when this interval is bounded from below (respectively, bounded from above) then the matrix $A + \alpha bc^T$ (respectively, $A + \beta bc^T$) has pure imaginary or zero eigenvalues.

In 1957 Kalman formulated the following conjecture.

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Kalman Conjecture. *If $S(\alpha,\beta)$ is a sector of linear stability for the system* (1) *and $\varphi:\mathbb{R}\to\mathbb{R}$ is continuously differentiable satisfying $\varphi'(\sigma)\in(\alpha,\beta)$ then the origin is a global attractor.*

We recall that an equilibrium point p is a *global attractor* for a differential system defined in $\mathbb{R}^n$ if the ω–limit of any orbit of the differential system is p. See for instance [8] for the definition of ω–limit of an orbit.

In 1988 Barabanov [2] proved the Kalman conjecture for $n=2,3$ and tried to provide a counterexample to Kalman conjecture in dimension 4 but his construction has some gaps. Bernat and Llibre [4] in 1996 presented the first counterexample to that conjecture in dimension larger than 3. For a clear proof of Kalman conjecture for $n=3$ see [7].

A C^1 differential system $\dot{\mathbf{x}}(t)=F(\mathbf{x}(t))$ defined in $\mathbb{R}^n$ is *Hurwitz* if the Jacobian matrix of $F(\mathbf{x})$ is Hurwitz at every point $\mathbf{x}\in\mathbb{R}^n$.

Consider a C^1 differential system $\dot{\mathbf{x}}(t)=F(\mathbf{x}(t))$ defined in $\mathbb{R}^n$ and having an equilibrium point at the origin of coordinates. If $DF(0)$ is Hurwitz then by Hartman–Grobman Theorem [17] the origin is locally asymptotically stable. A natural question is: *what hypotheses we have to add to the function $F(x)$ in order to assure that the origin is a global attractor*. Markus and Yamabe [23] in 1960 made the next conjecture.

Markus–Yamabe conjecture. *If we have a C^1 Hurwitz differential system $\mathbf{x}'=F(\mathbf{x})$ defined in $\mathbb{R}^n$ and having a unique equilibrium point at the origin of coordinates, then the origin is a global attractor.*

The Markus-Yamabe conjecture for $n=1$ follows easily. Some authors proved the Markus-Yamabe conjecture adding additional assumptions. For instance, Markus and Yamabe in [23] proved the conjecture if $F=(F_1,F_2)$ and $\partial F_i/\partial x_j=0$ for $i\neq j$, $i,j\in\{1,2\}$. In 1963 the conjecture was proved by Olech in [25] if $\int_0^\infty \min_{||x||=r}||F(x)||dr=\infty$. Gasull, Llibre and Sotomayor in [14] provided a list with more than ten different additional sufficient conditions forcing the Markus–Yamabe conjecture for $n=2$. Meistres and Olech [24] in 1988 proved the Markus-Yamabe conjecture for $n=2$ when F_1 and F_2 are polynomials.

Finally the conjecture was proved without additional hypotheses in independently by Gutierrez [18, 19] and by Fessler [9, 10]. Gutierrez had the proof in 1992 but it was published in 1993, and Fessler seems prove it in 1993, but it was published in 1995. In 1994 a simpler proof was provided by Glutsyuk [15, 16].

Also there are some additional assumptions forcing the Markus–Yamabe conjecture when $n>2$. Thus in 1961 the conjecture was proved by Hartman in [20] if $DF(x)$ is negative definite for all $x \in \mathbb{R}^n$. Other additional sufficient conditions, under which the conjecture holds, were given in 1962 by Hartamn and Olech in [21].

The counterexample to the Kalman conjecture for $n > 3$ given in [4], also is a counterexample to Markus–Yamabe conjecture for $n > 3$. Cima, van den Essen, Gasull, Hubbers and Mañosas provided a counterexample to the Markus–Yamabe conjecture for $n = 3$ given by a polynomial differential system.

In summary, now we know that the Markus–Yamabe conjecture holds in $\mathbb{R}^2$, but does not hold in $\mathbb{R}^n$ for all $n > 2$.

We can say that the study of the continuous or discontinuous piecewise linear differential systems started with Andronov, Vitt and Khaikin in [1]. After these systems became a topic of great interest in the mathematical community due to their applications in many areas, because they are used for modeling real phenomena and different modern devices, see for instance the books [3, 26] and references quoted therein.

Here we extend the Markus–Yamabe conjecture to continuous and discontinuous piecewise linear differential systems formed by two pieces of $\mathbb{R}^n$ separated by a hyperplane. More precisely, without loss of generality we consider the following class of piecewise linear differential systems in $\mathbb{R}^n$

$$\dot{\mathbf{x}} = \begin{cases} A^+\mathbf{x} + b^+ & \text{if } x_1 \geq 0, \\ A^-\mathbf{x} + b^- & \text{if } x_1 \leq 0. \end{cases} \tag{2}$$

If $A^+\mathbf{x} + b^+ = A^-\mathbf{x} + b^-$ in all points $\mathbf{x} = (0, x_2, \dots, x_n)$, then we say that the differential system (2) is a *continuous piecewise linear differential system*; otherwise we say that it is a *discontinuous piecewise linear differential system*. We note that the dynamics of this discontinuous differential system on the straight line of discontinuity is defined according with the definitions of the book of Filippov [11].

Since we will study the global stability of system (2) at an equilibrium point, it forces us to consider the case that the systems $\dot{\mathbf{x}} = A^+\mathbf{x} + b^+$ and $\dot{\mathbf{x}} = A^-\mathbf{x} + b^-$ both have a unique equilibrium point. So we will consider without loss of generality that A^+ and A^- are both invertible.

We say that the linear differential system $\dot{\mathbf{x}} = A^+\mathbf{x} + b^+$ has a *real equilibrium point* if the equilibrium point $-(A^+)^{-1}b^+$ exists, and it is in the closed half-space $\{x_1 \geq 0\}$, otherwise that equilibrium point is called *virtual*. Similarly the linear differential system $\dot{\mathbf{x}} = A^-\mathbf{x} + b^-$ has a *real equilibrium point* if the equilibrium point $-(A^-)^{-1}b^-$ exists, and it is in the closed half-space $\{x_1 \leq 0\}$, otherwise that equilibrium point is called *virtual*.

A *Markus–Yamabe piecewise linear differential system* is a differential system (2) such that the matrices A^+ and A^- are Hurwitz, and either only one of the systems $\dot{\mathbf{x}} = A^+\mathbf{x} + b^+$ and $\dot{\mathbf{x}} = A^-\mathbf{x} + b^-$ has a real equilibrium point, or both systems have the same real equilibrium point in $\{x_1 = 0\}$.

The objective of this paper is to answer the following question: *Determine the values of $n \geq 2$ for which all continuous (respectively discontinuous) Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^n$ have a global attractor in their real equilibrium point.* That is, our goal is to consider the conjecture of Markus–Yamabe made for C^1 differential systems to continuous and discontinuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^n$.

Our main results are the following two theorems, which characterize when the extension of the Markus–Yamabe conjecture holds or not for continuous and discontinuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^n$.

Theorem 1. *The following statements hold.*

(a) *The equilibrium point of all the continuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^2$ is a global attractor.*
(b) *For all $n > 2$ there are continuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^n$ for which their equilibrium point is not a global attractor.*

Theorem 1 is proved in section 2. A big part of this proof uses results of the articles [6] and [12].

We note that the answer to the extended Markus–Yamabe conjecture for continuous piecewise linear differential systems is the same than for the C^1 differential systems. But this is not the case for the discontinuous piecewise linear differential systems as it is shown in the following result.

Theorem 2. *For all $n \geq 2$ there are discontinuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^n$ for which their equilibrium point is not a global attractor.*

Theorem 2 is proved in section 2.

We note that for a very special class of discontinuous piecewise linear differential systems in $\mathbb{R}^2$, different from the classes here studied, the extension of the Markus–Yamabe conjecture to them holds, see Llibre and Teixeira [22].

2. Proof of the results

In order to prove Theorem 1 we shall use the canonical forms of the piecewise linear differential systems (2) provided in [13].

We consider the piecewise linear differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2\ell & -1 \\ \ell^2-\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix}, \tag{3}$$

defined in $\{x \leq 0\}$, and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2r & -1 \\ r^2-\beta^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix}, \tag{4}$$

defined in $\{x \geq 0\}$, where $\alpha, \beta \in \{i, 0, 1\}$, being i is the imaginary unit. If $\alpha = i$ then the equilibrium point of system (3) has eigenvalues $\ell \pm i$, so it is a focus if $\ell \neq 0$, and a center if $\ell = 0$. If $\alpha = 0$ then system (3) is a node with eigenvalue $\ell \neq 0$ of multiplicity 2 whose linear part does not diagonalize. If $\alpha = 1$ then system (3) is a saddle with eigenvalues $\ell - 1$ and $\ell + 1$ when $|\ell| < 1$, and a node with eigenvalues $\ell - 1$ and $\ell + 1$ whose linear part diagonalize when $|\ell| > 1$.

Let U be an open subset of $\mathbb{R}^2$. We say that the homeomorphism h between U and its image by h is a *topological equivalence* between the piecewise linear differential system (2) and the piecewise linear differential system (3)+(4) if h applies orbits of system (2) contained in U into orbits of system (3)+(4) contained in $h(U)$.

From Propositions 1 and 2 of [13] it follows that there exists a topological equivalence between the phase portrait of the piecewise linear differential system (2) and the phase portrait of the piecewise linear differential system (3)+(4) restricted to the orbits that do not have points in common with the sliding set of these systems. Therefore, since we are interested in studying when the unique equilibrium point of a Markus–Yamabe piecewise linear differential system (2) is a global attractor and the sliding set for our piecewise linear differential systems is at most formed by one orbit we do not need to take care of the sliding set. For a definition of the sliding set see [11].

Since we shall use the canonical forms for studying the continuous piecewise linear differential systems, we can restricted the canonical forms (3)+(4) to the continuous canonical forms on $x = 0$, i.e. to the continuous piecewise linear differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2\ell & -1 \\ \ell^2 - \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix}, \tag{5}$$

defined in $\{x \le 0\}$, and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2r & -1 \\ r^2 - \beta^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix}, \tag{6}$$

defined in $\{x \ge 0\}$.

Proof of Theorem 1. Statement (a) of Theorem 1 when both linear differential systems of (2) have the same real equilibrium point in $\{x = 0\}$ is proved in [12]. Now we shall prove statement (a) when only one of the two linear differential systems of (2) has a real equilibrium point, without loss of generality we can assume that this real equilibrium point is $-(A^+)^{-1}b^+$. So we can assume that the continuous piecewise linear differential systems (5)+(6) has a unique equilibrium point in the open half–plane $\{x > 0\}$, and we must prove that this equilibrium point is a global attractor.

We claim that the straight line $x = 0$ is tranversal under the flow of the continuous piecewise linear differential systems (5)+(6) except at the origin. Recall that a line segment is *transversal* under the flow of systems (5)+(6) if it has no contact points with the flow of systems (5)+(6), and that a *contact point* p of $x = 0$ is a point where the vector field $(2\ell x - y, (\ell^2 - \alpha^2)x + a)$ associated to systems (5)+(6) at p is parallel to the straight line $x = 0$. That is, if $p = (0, y_0)$ is a contact point of a system (5)+(6) with the straight line $x = 0$, then the inner product

$$(2\ell x - y, (\ell^2 - \alpha^2)x + a) \cdot (1, 0)|_{(x,y)=(0,y_0)} = -y_0.$$

This means that system (5)+(6) has a unique contact point with the straight line $x = 0$, which is at the origin $(0, 0)$.

Since by assumptions the matrices of the continuous piecewise linear differential systems (5)+(6) are Hurwitz, the equilibrium point contained in the open half–plane $\{x > 0\}$ is a local attractor. Therefore on the straight line $x = 0$ the flow of systems (5)+(6) enters in forward time at the half–plane $\{x > 0\}$. Hence, since systems (5)+(6) restricted to $\{x > 0\}$ is a linear differential system, all the orbits contained in $\{x \ge 0\}$ have ω–limit the equilibrium point contained in $\{x > 0\}$.

On the other hand, since systems (5)+(6) are Markus-Yamabe in the half–plane $\{x < 0\}$ they do not have equilibria, the equilibrium point of a linear system (5) is virtual, i.e. it is in $\{x > 0\}$ which is also a global attractor for the orbits of the linear system (5) in $\mathbb{R}^2$, so all the orbits of a system (5)+(6) starting at a point of $\{x < 0\}$ cross the straight line $x = 0$ and enter in the open half–plane $\{x > 0\}$, and consequently the equilibrium point contained in $\{x > 0\}$ is a global attractor. This concludes the proof of statement (a).

Statement (b) of Theorem 1 follows directly from Theorem 1 of [6]. More precisely, in [6] the authors proved the existence of continuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^3$ having a unique unstable equilibrium point at the origin of coordinates. In fact, the origin has an one-dimensional stable manifold and a two-dimensional invariant manifold, which is an attractive cone, on which the dynamics can be of stable or unstable focus type, or of center type.

Of course, these examples of continuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^3$ for which the origin is unstable, can be extended to continuous Markus–Yamabe piecewise linear differential systems in $\mathbb{R}^n$ with $n > 3$ adding to the 3–dimensional differential system the equations $\dot{x}_k = -x_k$ for $k = 4, \ldots, n$. □

Proof of Theorem 2. It is sufficient to prove the theorem for $n = 2$, because we can extend a discontinuous Markus–Yamabe piecewise linear differential system in $\mathbb{R}^2$ for which the unique equilibrium point of the system is not a global attractor, to a discontinuous Markus–Yamabe piecewise linear differential system in $\mathbb{R}^n$ with $n > 2$ for which its unique equilibrium point will not be a global attractor, adding to the 2–dimensional system the equations $\dot{x}_k = -x_k$ for $k = 2, \ldots, n$.

First we consider the discontinuous piecewise linear differential system in $\mathbb{R}^2$ defined by

$$
\begin{aligned}
&\dot{x} = y - 1, \quad \dot{y} = 1 - x, \quad \text{in } x \le 0, \\
&\dot{x} = y, \qquad\quad \dot{y} = 1 - x, \quad \text{in } x \ge 0.
\end{aligned} \tag{7}
$$

Note that this discontinuous piecewise linear differential system is formed by two linear centers. Moreover the orbits in the half-plane $x \le 0$ are contained in the arcs of the circles $\{(x-1)^2 + (y-1)^2 = r^2\} \cap \{x \le 0\}$ with $r \ge 1$, while the orbits in the half-plane $x \ge 0$ are contained in the arcs of the circles $\{(x-1)^2 + y^2 = r^2\} \cap \{x \ge 0\}$ with $r \ge 0$.

Then the orbit of the discontinuous piecewise linear differential system (7) starting at the point $(0, 2)$ crosses in forward time the straight line $x = 0$ by the first time at the point $(0, -2)$, by the second time at the point $(0, 4)$, by the third time at the point $(0, -4)$, by the fourth time at the point $(0, 6)$, ... So this orbit in forward time escapes to infinity spiraling intersecting the straight line $x = 0$ at the points $(0, \pm 2k)$ for all $k = 1, 2, \ldots$

Now we perturb slightly the discontinuous piecewise linear differential system (7) as follows

$$\begin{aligned} \dot{x} &= y - 1 - \varepsilon x, \quad \dot{y} = 1 - x, \quad \text{in } x \le 0, \\ \dot{x} &= y - \varepsilon x, \qquad\quad \dot{y} = 1 - x, \quad \text{in } x \ge 0, \end{aligned} \tag{8}$$

with $\varepsilon > 0$ sufficiently small.

Note that the two matrices A^+ and A^- of system (8) are Hurwitz. So system (8) is a discontinuous Markus–Yamabe piecewise linear differential system having the unique real equilibrium point $(1, \varepsilon)$, which is a stable focus of the right subsystem. Since $\varepsilon > 0$ is sufficiently small the orbit starting at the point $(0, 2)$ of system (7) which escapes spiraling to infinity, now for system (8) continues escaping to infinity. Consequently the equilibrium point $(1, \varepsilon)$ is not a global attractor. This completes the proof of the theorem. $\square$

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References

[1] A. Andronov, A. Vitt and S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.

[2] N.E. Barabanov, *On the Kalman Problem*, Sibirskii Matematicheskii Zhurnal **29** (1988), 3–11.

[3] M. Di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, *Piecewise–Smooth Dynamical Systems: Theory and applications*, Appl. Math. Sci. Series **163**, Springer–Verlag, London, 2008.

[4] J. Bernat and J. Llibre, *Counterexample to Kalman and Markus–Yamabe conjectures in dimension larger than* 3, Dynam. Contin. Discrete Impuls. Systems **2** (1996), 337–379.

[5] B. Brogliato, R. Lozano, B. Mashchke and O. Egeland, *Dissipative Systems Analysis and Control. Theory and Applications*, Second edition, Communications and Control Engineering Series, Springer–Verlag London, Ltd., London, 2007, pp 87–91.

[6] V. Carmona, E. Freire, E. Ponce and F. Torres, *The continuous matching of two stable linear systems can be unstable*, Discrete Contin. Dyn. Syst. **16** (2006), 689–703.

[7] A. Cima, A. van den Essen, E.Hubbers and F. Mañosas, *A polynomial counterexample to the Markus–Yamabe conjecture*, Adv. Math. **131** (1997), 453–457.

[8] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative Theory of Planar Differential Systems*, Springer Verlag, New York, 2006.

[9] R. Fessler, *On the Markus–Yamabe conjecture*, in Automorphisms of affine spaces (Curaçao, 1994), Kluwer Acad. Publ., Dordrecht, 1995, pp 127–135.

[10] R. Fessler, *A proof of the two–dimensional Markus–Yamabe stability conjecture and a generalization*, Ann. Polon. Math. **62** (1995), 45–74.

[11] A.F. Filippov, *Differential Equations with Discontinuous Right–Hand Sides*, Translated from the Russian, Mathematics and its Applications (Soviet Series) **18**, Kluwer Academic Publishers Group, Dordrecht, 1988.

[12] E. Freire, E. Ponce, F. Rodrigo and F. Torres, *Bifurcation sets of continuous piecewise linear systems with two zones*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **8** (1998), 2073–2097.

[13] E. Freire, E. Ponce and F. Torres, *A general mechanism to generate three limit cycles in planar Filippov systems with two zones*, Nonlinear Dynamics **78** (2014), 251–263.

[14] A. Gasull, J. Llibre and J. Sotomayor, *Global asymptotic stability of differential equations in the plane*, J. Differential Equations **91** (1991), 327–335.

[15] A.A. Glutsyuk, *A complete solution of the Jacobian problem for vector fields on the plane*, Russian Math. Survey **49** no. 3 (1994), 185–186.

[16] A.A. Glutsyuk, *The asymptotic stability of the linearization of a vector field on the plane with a singular point implies global stability*, (Russian) Funktsional. Anal. i Prilozhen. **29** (1995), no. 4, 17–30; translation in Funct. Anal. Appl. **29** (1996), no. 4, 238C-247.

[17] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer–Verlag, New York, 1983.

[18] C. Gutierrez, *A solution to the bidimensional global asymptotic stability conjecture*, conference given at the Workshop: Recent Results on the Global Asymptotic Stability Jacobian Conjecture, 1993.

[19] C. Gutierrez, *A solution to the bidimensional global asymptotic stability conjecture*, Ann. Inst. H. Poincar Anal. Non Linaire **12** (1995), no. 6, 627–671.

[20] P. Hartmann, *On stability in the large for systems of ordinary differential equations*, Canadian J. Math. **13** (1961), 480–492.

[21] P. Hartman and O. Olech, *On the global asymptotic stability of differential equations*, Trans. Amer. Math. Soc. **104** (1962), 154–178.

[22] J. Llibre and M.A. Teixeira, *Global asymptotic stability for a class of discontinuous vector fields in* $\mathbb{R}^2$, Dyn. Syst. **22** (2007), 133–146.

[23] L. Markus and H. Yamabe, *Global stability criteria for differential systems*, Osaka Math. J. **12** (1960), 305–317.

[24] G. Meisters and O. Olech, *Solution of the global asymptotic stability Jacobian conjecture for the polynomial case*, Analyse Mathématique et Applications, Contributions in l'honneur of J.L. Lions, Gauthier–Villars, Paris, 1988, pp. 373–381.

[25] O. Olech, *On the global stability of an autonomous system on the plane*, Contributions to Differential Equations **1** (1963), 389–400.

[26] D.J.W. Simpson, *Bifurcations in Piecewise–Smooth Continuous Systems*, World Scientific Series on Nonlinear Science A, vol **69**, World scientific, Singapure, 2010.

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