A SURVEY ON ALGEBRAIC AND EXPLICIT NON-ALGEBRAIC LIMIT CYCLES IN PLANAR DIFFERENTIAL SYSTEMS

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Abstract. In the qualitative theory of differential equations in the plane one of the most difficult objects to study is the existence of limit cycles. There are many papers dedicated to this subject. Here we will present a survey mainly dedicated to the algebraic and explicit non-algebraic limit cycles of the polynomial differential systems in $\mathbb{R}^2$ and of the discontinuous piecewise differential systems in $\mathbb{R}^2$ formed by two linear differential systems separated by a straight line. For this class of discontinuous piecewise differential systems the study of their algebraic and explicit non-algebraic limit cycles just is starting. Here we provide the first explicit non-algebraic limit cycle for the discontinuous piecewise linear differential systems. Additionally we recall seven open questions related with these types of limit cycles.

1. Introduction

We start by recalling the definition of the two classes of differential systems whose algebraic and explicit non-algebraic limit cycles we will study.

Let $P(x,y)$ and $Q(x,y)$ be real polynomials in the variables x and y. Then the differential system

$$\begin{aligned} \dot{x} &= P(x,y),\\ \dot{y} &= Q(x,y), \end{aligned} \tag{1}$$

where as usual the dot denotes the derivative with respect to the independent variable t, is a *polynomial differential system*. The maximum of the degrees of the polynomials P and Q is the *degree* of the polynomial differential system (1).

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Here we consider the discontinuous piecewise differential systems

$$X^{\pm} : (\dot{x}, \dot{y}) = (f^{\pm}(x, y), g^{\pm}(x, y)), \tag{2}$$

defined in the half-planes $\Sigma^{\pm} = \{(x, y) \in \mathbb{R}^2 : \pm x > 0\}$. On the straight line $\Sigma = \{x = 0\}$ the differential system is bivaluated. The straight line Σ is called the *straight line of discontinuity* when the two vector fields $X^{\pm}$ do not coincide on it. We use the Filippov conventions for defining the discontinuous piecewise differential system on Σ, see [33]. If $f^{+}(0, y)f^{-}(0, y) > 0$ at the point $(0, y) \in \Sigma$ we say that $(0, y)$ is a *crossing point*. If a periodic orbit of a discontinuous piecewise differential system (2) has exactly two crossing points we say that it is a *crossing periodic orbit*.

A *limit cycle* (respectively *crossing limit cycle*) of system (1) (respectively (2)) is an isolated periodic orbit in the set of all periodic orbits (respectively crossing periodic orbits) of system (1) (respectively (2)).

Here if $h(x, y)$ is a real polynomial *irreducible* in the ring $\mathbb{R}[x, y]$ of all real polynomials in the variables x and y, the zero set $\{h(x, y) = 0\}$ is an *algebraic curve*. An *algebraic limit cycle* is a limit cycle contained in an algebraic curve of the plane, otherwise such a limit cycle is called *non–algebraic*. The *degree* of an algebraic limit cycle is the degree of the irreducible polynomial which defines the algebraic curve containing the limit cycle. It is well known that the orbits of a polynomial differential system (1) are contained in analytic curves, which usually are not algebraic curves.

In general it is a difficult problem to distinguish if a limit cycle is algebraic or not. The proof that the famous limit cycle exhibited in the van der Pol equation in 1926 (see [80]) was not algebraic arrived in 1995, see Odani [76]. The differential equation of van der Pol can be written as a polynomial differential system (1) of degree 3, but we do not know explicitly its limit cycle. More precisely, we do not know the explicit expression of the analytic curve which contains the non-algebraic limit cycle of van der Pol equation. We remark that all the algebraic limit cycles are *explicit* because we only know them when we provide the algebraic curve containing the limit cycle.

A crossing limit cycle of the piecewise differential system (2) is *algebraic* if all its points, with the exception of the ones which are in Σ, are contained in algebraic curves of the half-planes $\Sigma^{\pm}$. The *degree* (n^{-}, n^{+}) of an algebraic crossing limit cycle is formed by the degree of the irreducible polynomials defining the algebraic curves which contain the crossing limit cycle, thus n^{-} (respectively n^{+}) is the degree of the irreducible polynomial defining the algebraic curve in Σ^- (respectively Σ^+) which contains the piece of the crossing limit cycle in Σ^- (respectively Σ^+).

2. Algebraic limit cycles of polynomial differential systems

2.1. **Some general results on algebraic limit cycles.** A nice result on algebraic limit cycles is the following one.

Theorem 1 (Bautin–Christopher–Dolov–Kuzmin Theorem)**.** *Assume that $f(x,y)=0$ is a non-singular algebraic curve of degree m, and $cx+dy+e=0$ a straight line which does not intersect any bounded component of the algebraic curve $f(x,y)=0$. Let a and b real numbers such that $ac+bd\neq 0$. Consider the polynomial differential system*

$$\dot{x}=af-(cx+dy+e)f_y, \qquad \dot{y}=bf+(cx+dy+e)f_x, \tag{3}$$

of degree m. Then every bounded component of the algebraic curve $f(x,y)=0$ is a hyperbolic algebraic limit cycle of the polynomial differential system (3)*. Furthermore the differential system* (3) *has no other limit cycles.*

Bautin in [5] proved a result similar to Theorem 1. But Bautin's paper contains a mistake which was solved in [26], and this result was generalized in [27]. The present statement of Theorem 1 is due to Christopher, see [24] where also there is a proof of it.

Let $f=f(x,y)$ be a real polynomial in the variables x and y. The algebraic curve $f(x,y)=0$ of $\mathbb{R}^2$ is an *invariant algebraic curve* of the polynomial differential system (1) if for some polynomial $K\in\mathbb{R}[x,y]$ we have

$$P\frac{\partial f}{\partial x}+Q\frac{\partial f}{\partial y}=Kf. \tag{4}$$

From (4) we note that the gradient $(\partial f/\partial x,\ \partial f/\partial y)$ of the curve on the points of the algebraic curve $f(x,y)=0$ is orthogonal to the vector field (P,Q), and consequently the vector field (P,Q) is tangent to the curve $f(x,y)=0$ in all its points. In other words the curve $f(x,y)=0$ is formed by orbits of the vector field (P,Q). This explains the name of invariant algebraic curve given to the algebraic curve $f(x,y)=0$, because the curve $f(x,y)=0$ is invariant under the flow of the vector field (P,Q).

Consider a polynomial differential system having a unique irreducible invariant algebraic curve, then the following result provides the maximum number of algebraic limit cycles that such differential system can have in function of the degree of that algebraic curve, for a proof see for instance [55].

Theorem 2. *Suppose that the algebraic curve $f(x, y) = 0$ of degree m is the unique irreducible invariant algebraic curve of a polynomial vector field (P, Q). Then the vector field (P, Q) can have at most $[(m-1)(m-2)/2] + 1$ algebraic limit cycles. Moreover choosing that $f(x, y) = 0$ has the maximal number of ovals for the irreducible algebraic curves of degree m, there exist vector fields X of degree m having exactly $[(m-1)(m-2)/2] + 1$ algebraic limit cycles.*

2.2. **Configurations of limit cycles via algebraic limit cycles.** Hilbert in 1900 and in the second part of his 16th problem (see [43]) proposed to provide a uniform upper bound for the maximum number of limit cycles of all polynomial differential systems of a given degree, additionally he also proposed to study the possible configurations or distributions of limit cycles in the plane for all polynomial differential systems. This last question has been solved in [64] using algebraic limit cycles.

A finite set $C = \{C_1, \ldots, C_n\}$ of disjoint simple closed curves in the plane satisfying $C_i \cap C_j = \emptyset$ for all $i \neq j$ is called a *configuration of limit cycles*.

Let $C = \{C_1, \ldots, C_n\}$ and $C' = \{C'_1, \ldots, C'_m\}$ be two configurations of limit cycles we say that they are (*topologically*) *equivalent* if there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h\left(\cup_{i=1}^n C_i\right) = \left(\cup_{i=1}^m C'_i\right)$. Of course, in order that two equivalent configurations of limit cycles C and C' be equivalent we need that $n = m$.

The configuration of limit cycles C is *realized* by a polynomial vector field (P, Q) if the set of all limit cycles of (P, Q) is equivalent to C.

In 2004 and in [64] it was proved the following result.

Theorem 3. *Let $C = \{C_1, \ldots, C_n\}$ be an arbitrary configuration of limit cycles. Then the configuration C is realizable with algebraic limit cycles by a polynomial vector field.*

Inspired in the proof of Theorem 3 it is possible to provide an alternative proof using the Bautin–Christopher–Dolov–Kuzmin Theorem.

In 2013 Coll, Dumortier and Prohens in [25] proved that any configuration of limit cycles can be realized for a convenient Liénard polynomial differential system.

2.3. **The 16th Hilbert problem restricted to algebraic limit cycles.** The original 16th Hilbert problem states: give an upper bound of the maximum number of limit cycles that the polynomial differential systems of degree n can exhibit in function of n, see more details on this famous problem in [43, 47, 50]. In this subsection we consider this problem restricted to the algebraic limit cycles.

If a set of irreducible algebraic curves, $f_j = f_j(x, y) = 0$ for $j = 1, \ldots, k$, satisfies the following five conditions it is called *generic.*

(i) The curve $f_j = 0$ is a non-singular algebraic curve, i.e. there are no points (x,y) satifying simultaneously $f_j = 0$, $\partial f_j/\partial x = 0$ and $\partial f_j/\partial y = 0$.
(ii) The homogeneous terms of higher order of f_j have no repeated factors.
(iii) If two distinc curves $f_i = 0$ and $f_j = 0$ intersect in a point of the affine plane, they intersect transversally at that point.
(iv) For every point of the affine plane at most pass two curves $f_j = 0$.
(v) The highest order homogeneous terms of two distinct polynomials f_i and f_j do not have common factors.

In the paper [62] it was proved the following result.

Theorem 4. *Assume that the set of all irreducible invariant algebraic curves of a polynomial differential system of degree $n \geq 2$ is generic. Then the maximum number of algebraic limit cycles of this system is at most $1 + (n-1)(n-2)/2$ if n is even, and $(n-1)(n-2)/2$ if n is odd. And there are polynomial differential systems of degree n reaching these upper bounds.*

Related with the results of Theorem 4 are the results of the paper [63].

In the papers [52, 55] appear the following open problem very related with Theorem 4.

Open problem 1. *Every quadratic polynomial differential system has at most one algebraic limit cycle.*

Partial results in the sense that the open problem 1 will have a positive answer are given in the papers [70, 71, 72].

From Theorem 4 it follows that if all the irreducible invariant algebraic curves of a cubic polynomial differential system are generic, then the system has at most 1 algebraic limit cycle. But there are cubic polynomial differential systems in [62] with two algebraic limit cycles, of course the invariant algebraic curves of those cubic polynomial differential systems are not generic. Thus the cubic polynomial differential system

$$\dot{x} = 2y(10 + xy), \quad \dot{y} = 20x + y - 20x^3 - 2x^2y + 4y^3,$$

has two algebraic limit cycles contained in the invariant algebraic curve $2x^4 - 4x^2 + 4y^2 + 1 = 0$, see Proposition 19 of [52]. Another cubic polynomial differential system having the two limit cycles $x^2 + y^2 = r^2$ and $(x - a)^2 + y^2 = r^2$ is

$$\begin{aligned} \dot{x} &= y(a^2 - r^2 - 3ax + 3x^2 - ay + 2xy + y^2), \\ \dot{y} &= -a^2x + 3ax^2 - 2x^3 - r^2y + axy - x^2y + y^3, \end{aligned}$$

when $r < a/2$.

These cubic polynomial differential systems exhibiting two algebraic limit cycles motivated that in the paper [55] appeared the next open problem, which also can be found in [52].

Open problem 2. *Every cubic polynomial differential system has at most two algebraic limit cycles.*

Clearly the open questions 1 and 2 hold for the quadratic and cubic polynomial differential systems if all their invariant algebraic curves are generic. But they remain open for those polynomial differential systems having non-generic invariant algebraic curves.

These two previous open problems where extended in [62] to the following most general open problem.

Open problem 3. *The maximum number of algebraic limit cycles that a polynomial differential system of degree n can have is $1 + (n - 1)(n - 2)/2$.*

The open problem 3 was proved for planar polynomial differential systems having only nodal invariant algebraic curves by Zhang in [86]

On the other hand the 16th Hilbert problem restricted to algebraic limit cycles formed by circles has been solved in [61].

2.4. **Algebraic limit cycles of quadratic polynomial differential systems.** The first algebraic limit cycle found in the quadratic polynomial differential systems is due to Qin [82, 83] and to Liu [53], they in 1957 and 1958 respectively proved that such systems can have algebraic limit cycles of degree 2, and that if a quadratic polynomial differential system has an algebraic limit cycle then it is the unique limit cycle of the system.

Later on it was proved that the quadratic polynomial differential systems cannot have algebraic limit cycles of degree 3 by Evdokimenco in [30, 31, 32], more recently this result has been proved in a shorter and clear way in [21, 54], and also in the projective quadratic polynomial differential systems by Zhang [87].

The first family of algebraic limit cycles of degree 4 in the quadratic polynomial differential systems was found in 1966 by Yablonskii [81]. In 1973 Filiptsov [34] found a second family of algebraic limit cycles of degree 4. In [21] two new families of algebraic limit cycles of degree 4 has been found, and the authors of [23] shown that the quadratic polynomial differential systems has no more families of algebraic limit cycles of degree 4. Moreover in [20] was proved that when a quadratic polynomial differential system has an algebraic limit cycle of degree 4 this is the unique limit cycle of the system. See [65, 66] for some other results on the algebraic limit cycles of the quadratic polynomial differential systems.

In [23] using birational transformation of the plane some families of algebraic limit cycles of degree 4 of the quadratic polynomial differential systems were transformed in new families of algebraic limit cycles of degrees 5 and 6 also for quadratic polynomial differential systems. These results were improved in [1] where a new family of algebraic limit cycles of degrees 5 was found for the quadratic polynomial differential systems.

Open problem 4. *Provide the maximum degree that can reach the algebraic limit cycles of the quadratic polynomial differential systems.*

Of course until now we know that the quadratic polynomial differential systems have algebraic limit cycles of degree 6, but it is unknown if these systems can have algebraic limit cycles of degree higher than 6.

3. Algebraic limit cycles of discontinuous piecewise linear differential systems

This is a very new area of research. At this moment we only know the paper of Buzzi, Gasull and Torregrosa [17] dedicated to this problem. In what follows we summarize the main results of [17].

For the family of piecewise linear differential systems (2) being $f^{\pm}(x,y)$ and $g^{\pm}(x,y)$ polynomials of degree 1 defined in $\Sigma^{\pm}$ the following statements hold.

(i) If a piecewise linear differential system (2) has an algebraic crossing limit cycle, then every linear differential system either has no equilibrium points, or has a unique equilibrium point of type: center, saddle, or node with distinct eigenvalues. Moreover, if the equilibrim point is a saddle or a node with distinct eigenvalues the quotient of both eigenvalues is a rational number.

(ii) If a piecewise linear differential system (2) has an algebraic crossing limit cycle, all the other limit cycles of the system are also algebraic, and all of them are nested. If Γ_1 and Γ_2 are two limit cycles, they are *nested* if either Γ_1 is contained in the region limited by Γ_2, or vice versa. In a similar way any number of limit cycles are said to be *nested* if every two of them are nested.

(iii) Does not exist an algebraic crossing limit cycles Γ of a piecewise linear differential system (2) such that the algebraic curve in Σ^- and the algebraic curve in Σ^+ containing Γ are defined by the same polynomial.

(iv) There are piecewise linear differential system (2) having exactly two algebraic crossing limit cycles.

(v) There are piecewise linear differential system (2) with exactly one semistable algebraic crossing limit cycle.

(vi) For every pair of positive integers (m,n) with $m \geq 2$ and $n \geq 2$ there exists a piecewise linear differential system (2) with an algebraic crossing limit cycle of degree (m,n).

In [17] the authors stated the following open problem.

Open problem 5. Can the piecewise linear differential system (2) exhibit more than two algebraic crossing limit cycles?

4. Explicit non-algebraic limit cycles of polynomial differential systems

Recently, since 2006 up to now, many articles have been showing explicit non-algebraic limit cycles in polynomial differential systems, i.e. in those articles the authors provided the explicit expression of the analytic curve containing the limit cycle. In this direction Gasull, Giacomini and Torregrosa [39] provided as far as we know the first explicit non-algebraic limit cycle for a polynomial differential system of degree 5. Clearly, if we multiply the right hand part of that quintic polynomial differential system by the expression $(ax+by+c)^n$ being n any given positive integer, and we choose the straight line $ax+by+c=0$ in such a way that it does not intersect the explicit non-algebraic limit cycle of the differential system, we obtain a new polynomial differential system of degree $5+n$ exhibiting an explicit non-algebraic limit cycle for all integer $n \geq 1$.

One year after the article of Gasull, Giacomini and Torregrosa and inspired in this article appeared the paper of Al-Dosary [2], where the author exhibits another explicit non-algebraic limit cycle for a distinct polynomial differential system of degree 5.

In 2006 Giné and Grau [41] shown the simultaneous existence of two limit cycles one algebraic and an explicit non-algebraic in a polynomial differential system of degree 9. Note that the paper [39] is quoted in [41].

The first paper providing an explicit non-algebraic limit cycle for polynomial differential systems of degree less than 5 was given by Benterki and Llibre [11] in 2012 for a polynomial differential system of degree 3, and of course it can be extended to any degree larger than 3.

Later on many other papers have been published providing explicit non-algebraic limit cycles for several polynomial differential systems, but in all these papers the explicit non-algebraic limit cycles are for polynomial differential systems of degree larger than or equal to 3, see [6, 7, 8, 9, 10, 11, 12, 13, 15, 16], the authors of these papers are Aziza, Bendjeddou, Benterki, Benyoucef, Berbache, Boukoucha, Boukoucha, Cheurfa, Grazem and Salhi.

Of course the polynomial differential systems of degree 1, i.e. the linear differential systems, cannot have limit cycles, because when they have a periodic orbit this forms part of a continuum of periodic orbits surrounding a center. In 2012 and in the paper [11] it was stated the following open problem, which remains open until now.

Open problem 6. *Provide an explicit non–algebraic limit cycle for a polynomial differential system of degree* 2.

We want to mention the paper of García [38] which provides a method for proving the non-algebraicity of limit cycles for particular polynomial systems under the generic assumption that the line at infinity is invariant.

5. Explicit non-algebraic limit cycles for the discontinuous piecewise linear differential systems

A discontinuous piecewise linear differential system with two pieces separated by a straight line in the plane $\mathbb{R}^2$ after a linear change of variables can be written into the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^- & a_{12}^- \\ a_{21}^- & a_{22}^- \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix} \text{ in } x < 0;$$

and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix} \text{ in } x > 0.$$

That is, without loss of generality we can assume that the discontinuity straight line is $x = 0$.

In 1930's Andronov, Vitt and Khaikin studied these discontinuous piecewise linear differential systems in their seminal book [3], where these differential systems appear in a natural way when they studied electrical, mechanical or control problems. The work of Levinson [48] is another important milestone in the history of the piecewise differential systems because it allowed to understand the complicated dynamics of the forced van der Pol equation, and this work inspired Smale [79] to find the horseshoe dynamics. Nowadays the piecewise linear systems continue receiving the attention of many researchers, see the books of di Bernardo [14] and Simpson [78], the survey of Makarenkov and Lamb [75], and the hundreds of references quoted in these last three works.

During these last twenty years many authors have been studied the limit cycles of the discontinuous piecewise linear differential systems (5), see for instance the papers [4, 18, 22, 19, 29, 35, 36, 37, 40, 42, 44, 45, 46, 51, 56, 57, 58, 60, 67, 68, 69, 74, 77, 84, 85]. The authors of these articles are Artés, Braga, Buzzi, Castillo, Ting Chen, Xiaoyan Chen, Euzébio, Freire, Giannakopoulos, Gouveia, Han, Chuangxia Huang, Lihong Huang, Wentao Huang, Liping Li, Llibre, Medrado, Mello, Novaes, Ordóñez, Pessoa, Pliete, Ponce, Rodrigo, Shui, Teixeira, Torregrosa, Torres, Verduzco, Wang, Jiazhong Yang, Xiao-Song Yang, Yu, Cheng Zhang, Weinian Zhang and Xiang Zhang. In all these papers at most 3 crossing limit cycles were found for the discontinuous piecewise linear differential systems (5), and from 2012 it remains the following open problem.

Open problem 7. *Is* 3 *the maximum number of crossing limit cycles that a discontinuous piecewise linear differential systems with a straight line of separation can have*?

In all these articles dedicated to study the crossing limit cycles of the discontinuous piecewise linear differential systems, do not appear explicit non-algebraic limit cycles, they proved their existence using different methods as the Poincaré map, the Melnikov function, the averaging theory, the Newton-Kantorovich Theorem, the first integrals, ...

In what follows we provide a new result showing an explicit non-algebraic limit cycle for the discontinuous piecewise linear differential system (2) defined by

$$\begin{aligned} f^-(x,y) &= \alpha x + y, \\ g^-(x,y) &= -x + \alpha y, \\ f^+(x,y) &= -\alpha x + y - \alpha, \\ g^+(x,y) &= -x - \alpha y - 1, \end{aligned} \tag{5}$$

with $\alpha \neq 0$.

It is easy to check that differential system (5) has the explicit non-algebraic crossing periodic orbit given by the analytic curves $(x^-(t), y^-(t))$ in the half-plane $x < 0$ for $t \in [0, \pi]$, and by $(x^+(t), y^+(t))$ in the half-plane $x > 0$ for $t \in [\pi, 3\pi/2]$ where

$$\begin{aligned} x^-(t) &= -e^{\alpha t - \frac{\pi\alpha}{2}} \sin t, \\ y^-(t) &= -e^{\alpha t - \frac{\pi\alpha}{2}} \cos t, \\ x^+(t) &= e^{-\alpha t} \left(\cos t - e^{-\frac{\pi\alpha}{2}} \sin t \right) - 1, \\ y^+(t) &= e^{-\alpha t} \left(-e^{-\frac{\pi\alpha}{2}} \cos t - \sin t \right). \end{aligned} \tag{6}$$

This non-algebraic crossing periodic orbit is drawn Figure 1 and has period $3\pi/2$. We shall provide the expression of the analytic curve containing the non-algebraic crossing periodic orbit parametrized by the time t, but using the first integrals of both linear differential systems and knowing that the non-algebraic crossing periodic orbit passes through the point $(0, -e^{-\pi\alpha/2})$ when $t = 0$ and through the point $(0, e^{\alpha\pi/2})$ when $t = \pi$. Thus these expressions are

$$\sqrt{x^2 + y^2} e^{\alpha \arctan\left(\frac{y}{x}\right)} = e^{\frac{\pi\alpha}{2}} \sqrt{e^{-\pi\alpha}},$$

in $x<0$, and

$$\frac{e^{4\alpha\arctan\left(\frac{x+\alpha y+1}{\alpha+\alpha x-y}\right)}}{\left((x+1)^2+y^2\right)^2}=\frac{e^{4\alpha\arctan\left(\frac{1-e^{-\frac{\pi\alpha}{2}}\alpha}{\alpha+e^{-\frac{\pi\alpha}{2}}}\right)}}{\left(e^{-\pi\alpha}+1\right)^2},$$

in $x>0$.

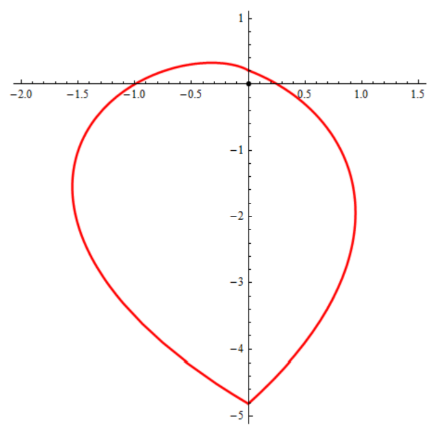

FIGURE 1. The non-algebraic limit cycle (6) of the piecewise linear differential system (5) for $\alpha=-1$.

Now we shall prove that this non-algebraic crossing periodic orbit is a limit cycle. For a smooth differential system (1) in the plane a periodic solution $(x(t),y(t))$ of period T such that

$$\int_0^T\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)(x(t),y(t))dt\neq 0, \tag{7}$$

is a limit cycle, usually called a *hyperbolic limit cycle*. This is a well known result, for a proof see for instance Theorem 1.23 of [28]. The formula (7) can be extended to the discontinuous piecewise linear differential systems here considered. Then for the discontinuous piecewise linear differential system (2)-(5) we have that

$$\int_0^{\pi}2\alpha dt+\int_{\pi}^{3\pi/2}(-2\alpha)dt=\alpha\pi\neq 0,$$

hence the non-algebraic crossing periodic orbit (6) is a limit cycle.

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