DARBOUX THEORY OF INTEGRABILITY IN $\mathbb{T}^n$

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

Abstract. We develop a Darboux theory of integrability for polynomial vector fields on the n-dimensional torus $\mathbb{T}^n$. Furthermore, we determine the maximum number of invariant parallels for a polynomial vector field on $\mathbb{T}^n$ depending on its degree.

1. Introduction

The Darboux theory of integrability [2] is one of the best tools to obtain first integrals for polynomial vector fields. It essentially builds a link between algebraic geometry and first integrals by showing that with a sufficient number of invariant algebraic surfaces, exponential factors and the multiplicity of the invariant algebraic surfaces, one can construct first integrals. The theory has been extended by many authors, starting with $\mathbb{R}^2$ and regular surfaces in $\mathbb{R}^3$, [9] and lately for $\mathbb{R}^n$, or $\mathbb{S}^n$, or either the Clifford torus [4, 5, 7, 6, 9, 8, 10, 11]. The importance of these extensions comes from the theory of first integrals and its applications. Their existence for differential systems is important in particular for the reduction of the ambient space, which in many cases makes easier the analysis of the dynamics. Our main aim is to obtain an extension of the Darboux theory of integrability of real polynomial vector fields on the n-dimensional torus $\mathbb{T}^n$, and then, using the extactic polynomial, to obtain the maximum number of parallels that such a vector field can have depending on its degree.

Before stating our results, we need some preliminary definitions (see [9] for more details).

Let $G\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^1 map. A hypersurface Ω defined by $G = 0$ is *regular* if $\nabla G \neq 0$ on Ω. We say that Ω is *algebraic of degree d* if G is an irreducible polynomial of degree d. A polynomial vector field X on Ω is a polynomial vector field X in $\mathbb{R}^{n+1}$ with

$$X \cdot \nabla G = 0 \quad \text{at all points of } \Omega.$$

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Given $f \in \mathbb{C}[x_1, ..., x_{n+1}]$, the algebraic hypersurface $\{f = 0\} \cap \Omega \subset \mathbb{R}^{n+1}$ is said to be *invariant* under a polynomial vector field X on Ω if there exists $k \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ (called the *cofactor* of $f = 0$ on Ω) such that $Xf = kf$ on Ω, and the two hypersurfaces $f = 0$ and Ω have transverse intersection.

Given $f, g \in \mathbb{C}_m[x_1, \ldots, x_{n+1}]$ (the set of all polynomials in $\mathbb{C}[x_1, \ldots, x_{n+1}]$ of degree at most m) and let $\Omega = \{G = 0\}$ be a regular algebraic hypersurface in $\mathbb{R}^{n+1}$ of degree d. We write $f \sim g$ if $f/g =$ constant or $f - g = hG$ for some polynomial h. One can verify that $\sim$ is an equivalence relation and we denote the dimension of the quotient space $\mathbb{C}_m[x_1, \ldots, x_{n+1}]/\sim$ by $d(m)$. It is proved in [9] that

$$d(m) = \binom{n+1+m}{n+1} - \binom{n+1+m-d}{n+1}. \tag{1}$$

Let $U \in \mathbb{R}^{n+1}$ be an open set. A real function $H(x_1, \ldots, x_{n+1}, t) \colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a *first integral of the polynomial vector field X on $\Omega \cap U$* if $H(x_1(t), \ldots, x_{n+1}(t))=$ constant for all the values of t such that $(x_1(t), \ldots, x_{n+1}(t)) \in \Omega \cap U$. If H is a rational function, then it is called a *rational first integral*.

Now we present the extension of the Darboux theory of integrability to polynomial vector fields on $\mathbb{T}^n$. The next theorem gives the dimension of $\mathbb{C}_m[x_1, \ldots, x_{n+1}]$.

Theorem 1. *We have*

$$d(m) = \binom{n+1+m}{n+1} - \binom{n+1+m-2^n}{n+1}.$$

Theorem 1 is proved in section 2. It follows from a general statement on the dimension of the linear space $\mathbb{C}_m[x_1, \ldots, x_{n+1}]$ on a regular algebraic hypersurface proved in [9]. We note that $m \geq 2^n$.

The following theorem follows readily from [9, Theorem 5] and [5, Theorem 2].

Theorem 2. *Let X be a polynomial vector field on $\mathbb{T}^n$ of degree $m = (m_1, \ldots, m_{n+1})$ having p invariant algebraic hypersurfaces $\{f_i = 0\} \cap \mathbb{T}^n$ with cofactors K_i for $i = 1, \ldots, p$ and q exponential factors $F_1, \ldots, F_q$ with $F_j = \exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$. Then the following statements hold:*

(a) *There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ on $\mathbb{T}^n$, if and only if the real (multi-valued) function of Darboux type $f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}$ substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbb{R}$ is a first integral of the vector field X on $\mathbb{T}^n$.*

(b) *If $p + q \geq d(m) + 1$ then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ on $\mathbb{T}^n$.*

(c) *There exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that* $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = -\sigma$ *on* $\mathbb{T}^n$ *for some* $\sigma \in \mathbb{R}\backslash\{0\}$ *if and only if the real (multi-valued) function of Darboux type* $f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} e^{\sigma t}$ *substituting* $f_i^{\lambda_i}$ *by* $|f_i|^{\lambda_i}$ *if* $\lambda_i \in \mathbb{R}$ *is an invariant of the vector field* X *on* $\mathbb{T}^n$.

(d) *The vector field* X *on* $\mathbb{T}^n$ *has a rational first integral if and only if* $p + q \geq d(m) + n$. *Moreover, all the trajectories are contained in invariant algebraic hypersurfaces.*

The *parallels* of the n-dimensional torus $\mathbb{T}^n$ are the intersections of $\mathbb{T}^n$ with the hyperplanes $x_1 =$ constant. Note that a parallel is a $(n-1)$-dimensional torus $\mathbb{T}^{n-1}$. An interesting question is how many invariant parallels a polynomial vector field in $\mathbb{T}^n$ can have depending on its degree m. The answer is given in the next theorem.

Theorem 3. *For* $n \geq 2$ *assume that* X *is a polynomial vector field on* $\mathbb{T}^n$ *of degree* $m = (m_1, \ldots, m_{n+1})$ *having finitely many invariant parallels. Then their number is at most* $\min\{m_1, \deg X - 2\}$.

Theorem 3 is proved in section 3. In particular, if $m_1 \geq m_2 \geq \cdots \geq m_{n+1}$, then $\deg X = m_1$ and so the maximum number of invariant parallels is $m_1 - 2$. In general, this upper bound is not reached and in any event the computations are very elaborate. But for the case of polynomial vector fields of degree four having $\mathbb{T}^2$ as an invariant algebraic surface we can go much further. Indeed, while in that case the upper bound on the maximum number of invariant parallels given by Theorem 3 is 2, we prove in the following theorems that the upper bound is 1 and we provide examples. The statement of this result is split into two theorems. In the first one we provide the most general form of all polynomial vector fields of degree four having $\mathbb{T}^2$ as an invariant algebraic surface and in the second one we prove that the maximum number of invariant parallels is 1.

Theorem 4. *Any polynomial vector field of degree* 4 *on* $\mathbb{T}^2$ *can be written in the form* $X = (P_1, P_2, P_3)$ *with* P_1, P_2, P_3 *given below in* (10), (11) *and* (12), *respectively.*

The proof of Theorem 4 is given in section 4.

Theorem 5. *There are no polynomial vector field of degree* 4 *on* $\mathbb{T}^2$ *having the maximum number of* 2 *invariant parallels. A polynomial vector field of degree* 4 *on* $\mathbb{T}^2$ *having one invariant parallel is* $X = (P_1, P_2, P_3)$ *with*

$$P_1 = 4(b_2c_0 - a_2d_0)(x_1 - \kappa)x_3(r_1^2 - r_2^2 - x_1^2 - x_2^2 - x_3^2),$$

$$\begin{aligned}P_2 = {} & 4x_3(b_1b_2c_0\kappa - a_1b_2d_0\kappa + (a_2b_1d_0 - b_1b_2c_0 + a_2b_1d_1\kappa - a_1b_2d_1\kappa)x_1 \\ & + (a_2b_1d_2\kappa - a_1b_2d_2)\kappa x_2 + (a_2b_1d_3\kappa - a_1b_2d_3\kappa)x_3)(x_1^2 + x_2^2 + x_3^2 - r_1^2 + r_2^2),\end{aligned}$$

and

$$
\begin{aligned}
P_3 &= b_2(a_2b_1 - a_1b_2)\kappa(r_1^2 - r_2^2)^2 - 4b_2(b_2c_0 - a_2d_0)\kappa(r_1^2 + r_2^2)x_1 + 2b_2(2b_2c_0 - 2a_2d_0 - a_2b_1\kappa \\
&\quad + a_1b_2\kappa)(r_1^2 + r_2^2)x_1^2 + 4b_2(b_2c_0 - a_2d_0)\kappa x_1^3 + b_2(4a_2d_0 - 4b_2c_0 + a_2b_1\kappa - a_1b_2\kappa)x_1^4 \\
&\quad + 4b_2(b_1c_0 - a_1d_0)\kappa(r_1^2 + r_2^2)x_2 - 4(a_1b_2d_1\kappa + b_1(b_2c_0 - a_2(d_0 + d_1\kappa)))(r_1^2 + r_2^2)x_1x_2 \\
&\quad + 4b_2(a_1d_0 - b_1c_0)\kappa x_1^2x_2 + 4(a_1b_2d_1\kappa + b_1(b_2c_0 - a_2(d_0 + d_1\kappa)))x_1^3x_2 - 2(a_2b_1 - a_1b_2)(b_2 \\
&\quad - 2d_2)\kappa(r_1^2 + r_2^2)x_2^2 + 4b_2(b_2c_0 - a_2d_0)\kappa x_1x_2^2 - 2(2a_2b_1d_2\kappa + b_2^2(2c_0 + a_1\kappa) - b_2(2a_2d_0 \\
&\quad + a_2b_1\kappa + 2a_1d_2\kappa))x_1^2x_2^2 + 4b_2(a_1d_0 - b_1c_0)\kappa x_2^3 + 4(a_1b_2d_1\kappa + b_1(b_2c_0 - a_2(d_0 + d_1\kappa)))x_1x_2^3 \\
&\quad + (a_2b_1 - a_1b_2)(b_2 - 4d_2)\kappa x_2^4 + 4(a_2b_1 - a_1b_2)d_3\kappa(r_1^2 + r_2^2)x_2x_3 + 4(a_1b_2 - a_2b_1)d_3\kappa x_1^2x_2x_3 \\
&\quad + 4(a_1b_2 - a_2b_1)d_3\kappa x_2^3x_3 + 2b_2(a_1b_2 - a_2b_1)\kappa(r_1^2 - r_2^2)x_3^2 + 4b_2(b_2c_0 - a_2d_0)\kappa x_1x_3^2 \\
&\quad + 2b_2(2a_2d_0 - 2b_2c_0 + a_2b_1\kappa - a_1b_2\kappa)x_1^2x_3^2 + 4b_2(a_1d_0 - b_1c_0)\kappa x_2x_3^2 + 4(a_1b_2d_1\kappa + b_1(b_2c_0 \\
&\quad - a_2(d_0 + d_1\kappa)))x_1x_2x_3^2 + 2(a_2b_1 - a_1b_2)(b_2 - 2d_2)\kappa x_2^2x_3^2 + 4(a_1b_2 - a_2b_1)d_3\kappa x_2x_3^3 \\
&\quad + b_2(a_2b_1 - a_1b_2)\kappa x_3^4.
\end{aligned}
$$

for any $\kappa, a_1, a_2, b_1, b_2, c_0, d_0, d_1, d_2, d_3 \in \mathbb{R}$ *with* $\kappa b_2 \neq 0$.

The proof of Theorem 5 is given in section 5.

We remark that the proofs and consequently the statements of Theorems 3, 4 and 5 depend on the parametrization chosen for $\mathbb{T}^n$, and consequently from the embedding of n-dimensional torus $\mathbb{T}^n$ in $\mathbb{R}^{n+1}$.

2. Proof of Theorem 1

We first introduce some preliminary results that will be used to prove Theorem 1.

Let $\mathbb{T}^n = (\mathbb{S}^1)^n$ be the n-dimensional torus in $\mathbb{R}^{n+1}$. We first define an embedding from $\mathbb{T}^n$ to $\mathbb{R}^{n+1}$. For this we consider the map $\Phi^{(n)} = \Phi^{(n)}_{r_1,\ldots,r_n} \colon (\mathbb{S}^1)^n \to \mathbb{R}^{n+1}$ given by $\Phi^{(n)}_{r_1,\ldots,r_n}(\alpha_1, \ldots, \alpha_n) = (x_1, \ldots, x_{n+1})$, where

$$
\begin{aligned}
x_1 &= r_1 \sin\alpha_1, \\
x_j &= \left(r_j + \frac{x_{j-1}}{\sin\alpha_{j-1}} \cos\alpha_{j-1}\right) \sin\alpha_j, \quad \text{for } j = 2, \ldots, n, \\
x_{n+1} &= \frac{x_n}{\sin\alpha_n} \cos\alpha_n
\end{aligned}
$$

with

$$
r_1 > 1 \quad \text{and} \quad r_j > \sum_{i=1}^{j-1} r_i \quad \text{for } j = 2, \ldots, n.
$$

Lemma 6. *The map* $\Phi = \Phi^{(n)}_{r_1,\ldots,r_n}$ *is injective.*

Proof. Let $\Phi^{(n-1)}_{r_1,\ldots,r_{n-1}}(\alpha_1,\ldots,\alpha_{n-1}) = (\bar{x}_1,\ldots,\bar{x}_n)$ be the former map of the torus $\mathbb{T}^{n-1}$ into $\mathbb{R}^n$. Note that we can write $\Phi^{(n)}_{r_1,\ldots,r_n}(\alpha_1,\ldots,\alpha_n) = (x_1,\ldots,x_{n+1})$ as follows:

$$
\begin{aligned}
x_j &= \bar{x}_j \quad \text{for } j = 1,\ldots,n-1,\\
x_n &= \Big(r_n + \frac{\bar{x}_n}{\sin\alpha_{n-1}}\cos\alpha_{n-1}\Big)\sin\alpha_n,\\
x_{n+1} &= \Big(r_n + \frac{\bar{x}_n}{\sin\alpha_{n-1}}\cos\alpha_{n-1}\Big)\cos\alpha_n.
\end{aligned}
$$

Now we shall prove the injectivity of $\Phi^{(n)}_{r_1,\ldots,r_n}$. We proceed by induction.

For $n = 2$ assume that $\Phi^{(2)}_{r_1,r_2}(\alpha_1,\alpha_2) = \Phi^{(2)}_{r_1,r_2}(\alpha_1',\alpha_2')$ with $\alpha_1,\alpha_2,\alpha_1',\alpha_2' \in [0,2\pi)$. Taking into account that

$$\Phi^{(2)}_{r_1,r_2}(\alpha_1,\alpha_2) = (r_1\sin\alpha_1, (r_2 + r_1\cos\alpha_1)\sin\alpha_2, (r_2 + r_1\cos\alpha_1)\cos\alpha_2),$$

we get

$$
\begin{aligned}
r_1\sin\alpha_1 &= r_1\sin\alpha_1',\\
(r_2 + r_1\cos\alpha_1)\sin\alpha_2 &= (r_2 + r_1\cos\alpha_1')\sin\alpha_2',\\
(r_2 + r_1\cos\alpha_1)\cos\alpha_2 &= (r_2 + r_1\cos\alpha_1')\cos\alpha_2'.
\end{aligned}
\tag{2}
$$

From the first equation we get $\sin\alpha_1 = \sin\alpha_1'$. From the second and third equations in (2) we get

$$(r_2 + r_1\cos\alpha_1)^2 = (r_2 + r_1\cos\alpha_1')^2.$$

Since $r_2 > r_1$, we have $\cos\alpha_1' = \cos\alpha_1$, and so $\alpha_1' = \alpha_1$, because $\sin\alpha_1 = \sin\alpha_1'$. Therefore, from the second and third equations in (2) we get $\sin\alpha_2 = \sin\alpha_2'$ and $\cos\alpha_2' = \cos\alpha_2$, respectively. Consequently $\alpha_2 = \alpha_2'$ which proves the claim for $n = 2$.

Now we assume that it holds until $n-1$ and we will prove it for n. Assume that

$$\Phi^{(n)}_{r_1,\ldots,r_n}(\alpha_1,\ldots,\alpha_n) = \Phi^{(n)}_{r_1,\ldots,r_n}(\alpha_1',\ldots,\alpha_n')$$

with $\alpha_1,\ldots,\alpha_n,\alpha_1',\ldots,\alpha_n' \in [0,2\pi)$. By the induction process and the construction of $\Phi^{(n)}_{r_1,\ldots,r_n}(\alpha_1,\ldots,\alpha_n)$ in terms of $\Phi^{(n-1)}_{r_1,\ldots,r_{n-1}}(\alpha_1,\ldots,\alpha_{n-1})$ we get $\alpha_1' = \alpha_1,\ldots,\alpha_{n-2}' = \alpha_{n-2}$ and

$$
\begin{aligned}
\frac{\bar{x}_n}{\sin\alpha_{n-1}}\sin\alpha_{n-1} &= \frac{\bar{x}_n}{\sin\alpha_{n-1}'}\sin\alpha_{n-1}',\\
\Big(r_n + \frac{\bar{x}_n}{\sin\alpha_{n-1}}\cos\alpha_{n-1}\Big)\sin\alpha_n &= \Big(r_n + \frac{\bar{x}_n}{\sin\alpha_{n-1}'}\cos\alpha_{n-1}'\Big)\sin\alpha_n',\\
\Big(r_n + \frac{\bar{x}_n}{\sin\alpha_{n-1}}\cos\alpha_{n-1}\Big)\cos\alpha_n &= \Big(r_n + \frac{\bar{x}_n}{\sin\alpha_{n-1}'}\cos\alpha_{n-1}'\Big)\cos\alpha_n'.
\end{aligned}
\tag{3}
$$

Note that since $\alpha_j = \alpha_j'$ for $j = 1, \dots, n-2$ and $\bar{x}_n / \sin \alpha_{n-1}$ only depends on $\alpha_1, \dots, \alpha_{n-2}$ we have that

$$\frac{\bar{x}_n}{\sin \alpha_{n-1}} = \frac{\bar{x}_n}{\sin \alpha_{n-1}'}. \tag{4}$$

From the second and third relations in (3) we get

$$\Big(r_n + \frac{\bar{x}_n}{\sin \alpha_{n-1}} \cos \alpha_{n-1}\Big)^2 = \Big(r_n + \frac{\bar{x}_n}{\sin \alpha_{n-1}'} \cos \alpha_{n-1}'\Big)^2,$$

and since $r_n > \sum_{j=1}^{n-1} r_j$ we readily obtain that

$$\frac{\bar{x}_n}{\sin \alpha_{n-1}} \cos \alpha_{n-1} = \frac{\bar{x}_n}{\sin \alpha_{n-1}'} \cos \alpha_{n-1}'.$$

Together with the first relation in (3) and using (4) we obtain

$$\cos \alpha_{n-1}' = \cos \alpha_{n-1} \quad \text{and} \quad \sin \alpha_{n-1}' = \sin \alpha_{n-1},$$

which yields $\alpha_{n-1} = \alpha_{n-1}'$. Now from the last two identities in (3) we obtain that $\alpha_n = \alpha_n'$ as we wanted to prove. This completes the proof of the lemma. $\square$

Now we continue with the proof of the theorem.

Using the parameterization $\Phi^{(n)}_{r_1,\dots,r_n}(\alpha_1, \dots, \alpha_n) = (x_1, \dots, x_{n+1})$ we obtain that the n-dimensional torus in cartesian coordinates can be expressed as follows:

$$y_{n+1}^2 + \varphi_n^2 = R_{n+1}^2, \quad y_j = x_{n+2-j},\ R_j = r_{n+2-j}, \quad j = 1, \dots, n+1, \tag{5}$$

and

$$\varphi_j = \sqrt{y_j^2 + \varphi_{j-1}^2} - R_j, \quad j = 2, \dots, n+1 \quad \text{and} \quad \varphi_1 = y_1. \tag{6}$$

We have the following lemma.

Lemma 7. *For each $n \geq 2$ there exists a polynomial $Q_{2^n} \in \mathbb{C}_{2^n}[x_1, \dots, x_{n+1}]$ of degree 2^n such that (5) can be written as*

$$Q_{2^n}(x_1, \dots, x_{n+1}) = 0.$$

Proof. We proceed by induction. If $n = 2$ then

$$y_3^2 + (\sqrt{y_2^2 + y_1^2} - R_2)^2 = R_3^2,$$

which can be written as

$$2R_2 \sqrt{y_2^2 + y_1^2} = R_3^2 - y_3^2 - y_2^2 - y_1^2 - R_2^2.$$

This yields

$$4R_2^2(y_1^2 + y_2^2) = (R_3^2 - y_3^2 - y_2^2 - y_1^2 - R_2^2)^2$$

or in other words

$$Q_4(y_1, y_2, y_3) = Q_4(x_1, x_2, x_3) = 0,$$

as we wanted to prove.

Now we assume that the statement holds until $n-1$ and we prove it for n. By the induction hypothesis we have that

$$y_n^2 + \varphi_{n-1}^2 - R_n^2 = Q_{2^{n-1}}(y_1, \dots, y_n).$$

It follows from (5) and (6) that

$$y_{n+1}^2 + y_n^2 + \varphi_{n-1}^2 + R_n^2 - 2R_n\sqrt{y_n^2 + \varphi_{n-1}^2} = R_{n+1}^2,$$

and so

$$y_{n+1}^2 + Q_{2^{n-1}}(y_1, \dots, y_n) + 2R_n^2 - R_{n+1}^2 = 2R_n\sqrt{y_n^2 + \varphi_{n-1}^2}.$$

Taking squares we get

$$\begin{aligned}(y_{n+1}^2 + Q_{2^{n-1}}(y_1, \dots, y_n) + 2R_n^2 - R_{n+1}^2)^2 &= 4R_n^2(y_n^2 + \varphi_{n-1}^2)\\ &= 4R_n^2(Q_{2^{n-1}}(y_1, \dots, y_n) + R_n^2),\end{aligned}$$

or in other words

$$Q_{2^n}(y_1, \dots, y_{n+1}) = 0 \quad \text{that is} \quad Q_{2^n}(x_1, \dots, x_{n+1}) = 0,$$

because $Q_{2^{n-1}}(y_1, \dots, y_n)^2$ is a polynomial of degree 2^n. This completes the proof of the lemma. □

It follows from Lemma 7 that $\mathbb{T}^n$ is regular and that we can rewrite it as

$$Q_{2^n}(x_1, \dots, x_{n+1}) = 0, \quad \text{for some polynomial } Q_{2^n} \text{ of degree } 2^n.$$

So we have that $d = 2^n$. Hence from (1) it follows that $\mathbb{C}_m[x_1, \dots, x_{n+1}]$ on the n-dimensional torus $\mathbb{T}^n$ is a $\mathbb{C}$-linear space of dimension

$$d(m) = \binom{m+n+1}{n+1} - \binom{m+n+1-2^n}{n+1}.$$

This completes the proof of the theorem.

3. Proof of Theorem 3

A convenient tool to look for invariant algebraic hypersurfaces is the extactic polynomial of X associated to a finitely generated vector subspace W of the vector space $\mathbb{C}[x_1, \dots, x_d]$ with basis $\{v_1, \dots, v_l\}$ (see for instance [4, 3, 12]). It is defined by

$$\mathcal{E}_W(X) = \mathcal{E}_{\{v_1,\dots,v_l\}}(X) = \det\begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ X(v_1) & X(v_2) & \cdots & X(v_l) \\ \vdots & \vdots & \vdots & \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \cdots & X^{l-1}(v_l)\end{pmatrix}.$$

In view of the properties of determinants, $\mathcal{E}_W(X)$ does not depend on the chosen basis of W. The next proposition is proved in [1].

Proposition 8. *Let X be a polynomial vector field in $\mathbb{C}^d$ and let W be a finitely generated vector subspace of $\mathbb{C}[x_1, \ldots, x_d]$ with $\dim W > 1$. Then every algebraic invariant hypersurface $f = 0$ for X, with $f \in W$, is a factor of $\mathcal{E}_W(X)$.*

It follows from Proposition 8 that $f = 0$ is an invariant hyperplane of X if the polynomial f is a factor of the polynomial $\mathcal{E}_W(X)$, where W is generated by $\{1, x_1, \ldots, x_d\}$.

Proof of Theorem 3. By definition an invariant parallel is the intersection of an invariant hyperplane of the form $x_1 = \kappa$, where $\kappa \in \mathbb{R}$, with the n-dimensional torus $\mathbb{T}^n$. Thus this intersection is a $\mathbb{T}^{n-1}$ $(n-1)$-dimensional torus. From Proposition 8 we know that if $x_1 - \kappa = 0$ is an invariant hyperplane of the polynomial vector field X, then $x_1 - \kappa$ is a factor of the extactic polynomial. So the maximum number of factors of the form $x_1 - \kappa$ of the extactic polynomial $\mathcal{E}_{\{1, x_{n+1}\}}(X)$ gives an upper bound for the number of invariant planes $\{x_1 - \kappa = 0\}$ of X, and this allows us to obtain an upper bound for the number of its invariant parallels.

From the definition of extactic polynomial we get

$$\det \begin{pmatrix} 1 & x_1 \\ X(1) & X(x_1) \end{pmatrix} = \det \begin{pmatrix} 1 & x_1 \\ 0 & P_1(x_1, \ldots, x_{n+1}) \end{pmatrix} = P_1 = P_1(x_1, \ldots, x_{n+1}).$$

Since the degree of P_1 is m_1, this polynomial can have at most m_1 linear factors of the form $x_1 - \kappa$ and so the number of invariant parallels of X on $\mathbb{T}^n$ is at most m_1.

However this bound can be improved after imposing that the n-dimensional torus $\mathbb{T}^n$ is an invariant algebraic hypersurface of the vector field $X = (P_1, \ldots, P_{n+1})$. First we recall that in view of Theorem 1 and its proof, we can write $\mathbb{T}^n$ as $F = 0$ being

$$F(x_1, \ldots, x_{n+1}) = \tilde{F}(x_1^2, \ldots, x_{n+1}^2) = \tilde{F}(z_1, \ldots, z_{n+1}),$$

and it has degree 2^n. Moreover it follows also from that theorem and its proof that

$$F(x_1, 0, \ldots, 0, 0) = x_1^2 + (r_{n+1} - \sum_{i=2}^{n} r_i)^2 - r_1$$

and $r_{n+1} > \sum_{i=2}^{n} r_i + r_1$ with $r_1 > 1$. Note that this implies that

$$\tilde{F}(x_1^2, 0, \ldots, 0, 0) = \tilde{F}(z_1, 0, \ldots, 0, 0) = z_1 + (r_{n+1} - \sum_{i=2}^{n} r_i)^2 - r_1.$$

Then

$$2x_1\frac{\partial \tilde{F}}{\partial z_1}P_1 + \ldots + 2x_{n+1}\frac{\partial \tilde{F}}{\partial z_{n+1}}P_{n+1} = K\tilde{F} \tag{7}$$

for all $(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1}$ where $K = K(x_1,\ldots,x_{n-1})$ is a polynomial of degree $m-1$ with $m = \deg X$.

We write

$$P_1 = h(x_1,\ldots,x_{n+1})\prod_{i=1}^{l}(x_1-\kappa_i),$$

in such a way that $x_1-\kappa_i$ for all $\kappa_i \in \mathbb{R}$ is not a factor of the polynomial h. Eventually some of the κ_i's can be the same. Then

$$\mathcal{E}_{\{1,x_1\}} = h(x_1,\ldots,x_{n+1})\prod_{i=1}^{l}(x_1-\kappa_i).$$

Since (7) holds for all $x_1,\ldots,x_{n+1} \in \mathbb{R}$, in particular it must hold for $x_2 = \cdots = x_{n+1} = 0$ and so,

$$2x_1 h(x_1,0,\ldots,0)\prod_{i=1}^{l}(x_1-\kappa_i) = \left(\sum_{i=0}^{m-1} k_i x_1^i\right)\left(x_1^2 + (r_{n+1}-\sum_{i=2}^{n} r_i)^2 - r_1\right),$$

where $k_i = k_i(x_2,\ldots,x_{n+1})$ is a polynomial for $i = 0,\ldots,m-1$. From this equation we have that $k_0 = 0$ and consequently

$$2x_1 h(x_1,0,\ldots,0)\prod_{i=1}^{l}(x_1-\kappa_i) = x_1\left(\sum_{i=0}^{m-2} k_i x_1^i\right)\left(x_1^2 + (r_{n+1}-\sum_{i=2}^{n} r_i)^2 - r_1\right).$$

Taking into account that $r_{n+1} > \sum_{i=2}^{n} r_i + r_1$ and $r_1 > 1$, we see that

$$x_1^2 + \left(r_{n+1} - \sum_{i=2}^{n} r_i\right)^2 - r_1 > x_1^2 + r_1^2 - r_1 > 0,$$

and consequently it does not factorize in $\mathbb{R}[x_1]$. This assertion together with the fact that $h(x_1,\ldots,x_{n+1})$ has no factor of the form $x_1-\kappa$ shows that $l \le m-2$. So $\mathcal{E}_{\{1,x_1\}}(X)$ has at most $m-2$ factors of the form $x_1 = \kappa$ with $\kappa \in \mathbb{R}$. Hence X has at most $m-2$ invariant hyperplanes of the form $x_1 = \kappa$ with $\kappa \in \mathbb{R}$, and consequently X has at most $m-2$ invariant parallels.

Therefore, the maximum number of invariant parallels that X can have is

$$\min\{m_1, m-2\} = \min\{m_1, \deg X - 2\}.$$

This completes the proof of the theorem. $\square$

4. Proof of Theorem 4

In view of Theorem 1 the n-dimensional torus $\mathbb{T}^2$ can be written in cartesian coordinates as the surface

$$g_1 = (x_1^2 + x_2^2 + x_3^2 + r_1^2 - r_2^2)^2 - 4r_2^2(x_1^2 + x_2^2) = 0, \tag{8}$$

which is the surface $Q_4(x_1, x_2, x_3) = 0$ of Lemma 7.

It follows from [7, Theorem 1.3.1] that any polynomial differential system in $\mathbb{R}^3$ having $g_1 = 0$ as an invariant algebraic surface must be written in the form $X = (P_1, P_2, P_3)$ where

$$\begin{aligned} P_1 &= \phi\{x_1, g_2, g_3\} + \lambda_1\{g_1, x_1, g_3\} + \lambda_2\{g_1, g_2, x_1\}, \\ P_2 &= \phi\{x_2, g_2, g_3\} + \lambda_1\{g_1, x_2, g_3\} + \lambda_2\{g_1, g_2, x_2\}, \\ P_3 &= \phi\{x_3, g_2, g_3\} + \lambda_1\{g_1, x_3, g_3\} + \lambda_2\{g_1, g_2, x_3\}, \end{aligned} \tag{9}$$

where ϕ is a polynomial in the variables x_1, x_2, x_3 satisfying $\phi|_{g_1=0} = 0$, and g_k for $k = 2, 3$ and λ_i for $i = 1, 2$ are arbitrary polynomials in the variables (x_1, x_2, x_3). Moreover $\{f, g, h\}$ denotes the *Nambu bracket* of the polynomials $f = f(x_1, x_2, x_3)$, $g = g(x_1, x_2, x_3)$, $h = h(x_1, x_2, x_3)$ which is defined as

$$\{f, g, h\} = \det \begin{pmatrix} f_{x_1} & f_{x_2} & f_{x_3} \\ g_{x_1} & g_{x_2} & g_{x_3} \\ h_{x_1} & h_{x_2} & h_{x_3} \end{pmatrix}.$$

Since we are looking for polynomial vector fields of degree four and g_1 is a polynomial of degree four, without loss of generality we can take $\phi = \phi(x_1, x_2, x_3) = g_1(x_1, x_2, x_3)$ (because rescaling the time if necessary any constant can be passed to one). Moreover since $\deg g_1 = 4$ we have that the degrees of $g_2, g_3, \lambda_1, \lambda_2$ must be one. So we take them as follows

$$\begin{aligned} g_2(x_1, x_2, x_3) &= a_0 + a_1x_1 + a_2x_2 + a_3x_3, \\ g_3(x_1, x_2, x_3) &= b_0 + b_1x_1 + b_2x_2 + b_3x_3, \\ \lambda_1(x_1, x_2, x_3) &= c_0 + c_1x_1 + c_2x_2 + c_3x_3, \\ \lambda_2(x_1, x_2, x_3) &= d_0 + d_1x_1 + d_2x_2 + d_3x_3, \end{aligned}$$

for any $a_i, b_i, c_i, d_i \in \mathbb{R}$ for $i = 0, \ldots, 3$.

It follows from (9) that P_1 is equal to

(10)

$$
\begin{aligned}
&-(a_3b_2-a_2b_3)(r_1^2-r_2^2)^2+4(b_3c_0-a_3d_0)(r_1^2+r_2^2)x_2-4(b_2c_0-a_2d_0)(r_1^2-r_2^2)x_3\\
&+2(a_3b_2-a_2b_3)(r_1^2+r_2^2)x_1^2+4(b_3c_1-a_3d_1)(r_1^2+r_2^2)x_1x_2-4(b_2c_1-a_2d_1)(r_1^2-r_2^2)x_1x_3\\
&+2(a_3b_2-a_2b_3+2b_3c_2-2a_3d_2)(r_1^2+r_2^2)x_2^2+4(a_2d_2r_1^2-a_3d_3r_1^2-a_2d_2r_2^2-a_3d_3r_2^2\\
&+b_2c_2(r_2^2-r_1^2)+b_3c_3(r_1^2+r_2^2))x_2x_3+2(a_3b_2-a_2b_3-2b_2c_3+2a_2d_3)(r_1^2-r_2^2)x_3^2\\
&+4(a_3d_0-b_3c_0)x_1^2x_2+4(b_2c_0-a_2d_0)x_1^2x_3+4(a_3d_0-b_3c_0)x_2^3+4(b_2c_0-a_2d_0)x_2^2x_3\\
&+4(a_3d_0-b_3c_0)x_2x_3^2+4(b_2c_0-a_2d_0)x_3^3+(a_2b_3-a_3b_2)x_1^4+4(a_3d_1-b_3c_1)x_1^3x_2\\
&+4(b_2c_1-a_2d_1)x_1^3x_3-2(a_3b_2-a_2b_3+2b_3c_2-2a_3d_2)x_1^2x_2^2+4(b_2c_2-b_3c_3-a_2d_2\\
&+a_3d_3)x_1^2x_2x_3+2(-a_3b_2+2b_2c_3+a_2(b_3-2d_3))x_1^2x_3^2+4(a_3d_1-b_3c_1)x_1x_2^3\\
&+4(b_2c_1-a_2d_1)x_1x_2^2x_3+4(a_3d_1-b_3c_1)x_1x_2x_3^2+4(b_2c_1-a_2d_1)x_1x_3^3-(a_3b_2-a_2b_3\\
&+4b_3c_2-4a_3d_2)x_2^4+4(b_2c_2-b_3c_3-a_2d_2+a_3d_3)x_2^3x_3+2(-a_3b_2+a_2b_3-2b_3c_2\\
&+2b_2c_3+2a_3d_2-2a_2d_3)x_2^2x_3^2+4(b_2c_2-b_3c_3-a_2d_2+a_3d_3)x_2x_3^3-(a_3b_2-4b_2c_3\\
&-a_2(b_3-4d_3))x_3^4,
\end{aligned}
$$

P_2 is equal to

(11)

$$
\begin{aligned}
&(a_3b_1-a_1b_3)(r_1^2-r_2^2)^2-4(b_3c_0-a_3d_0)(r_1^2+r_2^2)x_1+4(b_1c_0-a_1d_0)(r_1^2-r_2^2)x_3\\
&-2(a_3b_1-a_1b_3+2b_3c_1-2a_3d_1)(r_1^2+r_2^2)x_1^2-4(b_3c_2-a_3d_2)(r_1^2+r_2^2)x_1x_2\\
&-4(a_1d_1r_1^2-a_3d_3r_1^2-a_1d_1r_2^2-a_3d_3r_2^2+b_1c_1(r_2^2-r_1^2)+b_3c_3(r_1^2+r_2^2))x_1x_3\\
&-2(a_3b_1-a_1b_3)(r_1^2+r_2^2)x_2^2+4(b_1c_2-a_1d_2)(r_1^2-r_2^2)x_2x_3-2(a_3b_1-a_1b_3-2b_1c_3\\
&+2a_1d_3)(r_1^2-r_2^2)x_3^2+4(b_3c_0-a_3d_0)x_1^3+4(a_1d_0-b_1c_0)x_1^2x_3+4(b_3c_0-a_3d_0)x_1x_2^2\\
&+4(b_3c_0-a_3d_0)x_1x_3^2+4(a_1d_0-b_1c_0)x_2^2x_3+4(a_1d_0-b_1c_0)x_3^3+(a_3b_1-a_1b_3+4b_3c_1\\
&-4a_3d_1)x_1^4+4(b_3c_2-a_3d_2)x_1^3x_2-4(b_1c_1-b_3c_3-a_1d_1+a_3d_3)x_1^3x_3+2(a_3b_1-a_1b_3\\
&+2b_3c_1-2a_3d_1)x_1^2x_2^2+4(a_1d_2-b_1c_2)x_1^2x_2x_3+2(2b_3c_1-a_1b_3-2b_1c_3+a_3(b_1-2d_1)\\
&+2a_1d_3)x_1^2x_3^2+4(b_3c_2-a_3d_2)x_1x_2^3+4(b_3c_3-b_1c_1+a_1d_1-a_3d_3)x_1x_2^2x_3+4(b_3c_2\\
&-a_3d_2)x_1x_2x_3^2+4(b_3c_3-b_1c_1+a_1d_1-a_3d_3)x_1x_3^3+(a_3b_1-a_1b_3)x_2^4+4(a_1d_2-b_1c_2)x_2^3x_3\\
&+2(a_3b_1-a_1b_3-2b_1c_3+2a_1d_3)x_2^2x_3^2+4(a_1d_2-b_1c_2)x_2x_3^3+(a_3b_1-a_1b_3-4b_1c_3\\
&+4a_1d_3)x_3^4,
\end{aligned}
$$

and P_3 is equal to

$$
\begin{aligned}
&-(a_2b_1-a_1b_2)(r_1^2-r_2^2)^2+4(b_2c_0-a_2d_0)(r_1^2+r_2^2)x_1-4(b_1c_0-a_1d_0)(r_1^2+r_2^2)x_2\\
&+2(a_2b_1-a_1b_2+2b_2c_1-2a_2d_1)(r_1^2+r_2^2)x_1^2-4(b_1c_1-b_2c_2-a_1d_1+a_2d_2)(r_1^2+r_2^2)x_1x_2\\
&+4(b_2c_3-a_2d_3)(r_1^2+r_2^2)x_1x_3+2(a_2b_1-a_1b_2-2b_1c_2+2a_1d_2)(r_1^2+r_2^2)x_2^2\\
&-4(b_1c_3-a_1d_3)(r_1^2+r_2^2)x_2x_3+2(a_2b_1-a_1b_2)(r_1^2-r_2^2)x_3^2+4(a_2d_0-b_2c_0)x_1^3\\
&+4(b_1c_0-a_1d_0)x_1^2x_2+4(a_2d_0-b_2c_0)x_1x_2^2+4(a_2d_0-b_2c_0)x_1x_3^2+4(b_1c_0-a_1d_0)x_2^3\\
&+4(b_1c_0-a_1d_0)x_2x_3^2+(a_1b_2-a_2b_1-4b_2c_1+4a_2d_1)x_1^4+4(b_1c_1-b_2c_2-a_1d_1+a_2d_2)x_1^3x_2\\
&+4(a_2d_3-b_2c_3)x_1^3x_3+2(a_1b_2-a_2b_1-2b_2c_1+2b_1c_2+2a_2d_1-2a_1d_2)x_1^2x_2^2+4(b_1c_3\\
&-a_1d_3)x_1^2x_2x_3+2(a_1b_2-a_2b_1-2b_2c_1+2a_2d_1)x_1^2x_3^2+4(b_1c_1-b_2c_2-a_1d_1+a_2d_2)x_1x_2^3\\
&+4(a_2d_3-b_2c_3)x_1x_2^2x_3+4(b_1c_1-b_2c_2-a_1d_1+a_2d_2)x_1x_2x_3^2+4(a_2d_3-b_2c_3)x_1x_3^3\\
&-(a_2b_1-4b_1c_2-a_1(b_2-4d_2))x_2^4+4(b_1c_3-a_1d_3)x_2^3x_3+2(2b_1c_2-a_2b_1+a_1(b_2-2d_2))x_2^2x_3^2\\
&+4(b_1c_3-a_1d_3)x_2x_3^3-(a_2b_1-a_1b_2)x_3^4.
\end{aligned}
\tag{12}
$$

5. Proof of Theorem 5

In view of Theorem 1 the 2-dimensional torus $\mathbb{T}^2$ in cartesian coordinates can be written as the surface $g_1=0$ with g_1 as in (8). It follows from the proof of Theorem 4 that any polynomial vector field $X=(P_1,P_2,P_3)$ of degree four having $\mathbb{T}^2$ as an invariant surface must be written as in (10)–(12). Now it follows from the proof of Theorem 3 and the definition of invariant parallel that in order to obtain the most general polynomial vector fields having the maximum number of parallels (which is at most two) we must have that the polynomial P_1 in (10) must be of the form

$$
\begin{aligned}
P_1=&(x_1-\kappa_1)(x_1-\kappa_2)(s_0+s_1x_1+s_2x_2+s_3x_3+s_4x_1^2+s_5x_1x_2+s_6x_1x_3\\
&+s_7x_2^2+s_8x_2x_3+s_9x_3^2),
\end{aligned}
$$

for some $\kappa_1,\kappa_2,s_i\in\mathbb{R}$ for $i=0,\dots,9$. Solving this equation for any κ_1,κ_2,s_i we get that the unique eventual solution is $s_0=\dots=s_9=0$ which is not possible because then $P_1=0$. So, there are no polynomial vector fields on $\mathbb{T}^2$ of degree 4 having two invariant parallels. The most general form for a polynomial vector field on $\mathbb{T}^2$ of degree four having one parallel is

$$
\begin{aligned}
P_1=&(x_1-\kappa)(s_0+s_1x_1+s_2x_2+s_3x_3+s_4x_1^2+s_5x_1x_2+s_6x_1x_3+s_7x_2^2\\
&+s_8x_2x_3+s_9x_3^2+s_{10}x_1^3+s_{11}x_1^2x_2+s_{12}x_1^2z+s_{13}x_1x_2^2+s_{14}x_1x_2x_3\\
&+s_{15}x_1x_3^2+s_{16}x_2^3+s_{17}x_2^2x_3+s_{18}x_2x_3^2+s_{19}x_3^3),
\end{aligned}
$$

for some $\kappa,s_i\in\mathbb{R}$ for $i=0,\dots,19$ with $(s_0,\dots,s_{19})\neq(0,\dots,0)$. Solving this equation we obtain many solutions. One of these solutions is the solution provided in the statement of the theorem.

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This paper has no data.

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jaume.llibreuab.cat

[2]∗ Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: cvalls@math.tecnico.ulisboa.pt