

# GLOBAL CENTERS OF THE GENERALIZED POLYNOMIAL LIÉNARD DIFFERENTIAL SYSTEMS

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**ABSTRACT.** The global centers started to be studied seriously at the end of the XX century. In the last decades, the generalized polynomial Liénard differential systems have been studied intensively. In this paper we characterize all the generalized polynomial Liénard differential systems having a global center at the origin. In particular we provide the explicit expressions of all the generalized polynomial Liénard differential systems of degree 3 having a global center at the origin, and the explicit expression of a generalized polynomial Liénard differential system of degree 5 having a global center at the origin.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We say that  $p$  is a *center* of a differential system in  $\mathbb{R}^2$  if there exists a neighborhood  $U$  of  $p$  such that  $U \setminus \{p\}$  is filled up with periodic orbits. The notion of center goes back to the works of Poincaré [12] and Dulac [5].

The maximal connected set of periodic orbits surrounding the center  $p$  and having  $p$  in its boundary is called the *period annulus* of the center  $p$ . Moreover we say that  $p$  is a *global center* if its period annulus is  $\mathbb{R}^2 \setminus \{q\}$ .

As far as we know, the study on global centers was initiated by a group of researchers, see Conti et al. [3,4,7]. Although some works on global centers appeared in the literature before Conti et al., global centers were not named or defined. In [7] it is proved that the polynomial differential systems having a global center must have odd degrees. In [3] it is proved that the linear differential centers are the unique centers of the polynomial differential systems which are rigid and global. ‘We recall that a center  $p$  is *rigid* if its differential system in polar coordinates with the origin at  $p$  has its angular velocity constant. Different results on the global centers were obtained in [4]. Other studies on global centers for polynomial differential systems can be found in [8, 9, 10, 13].

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In this paper we deal with the real generalized polynomial Liénard differential equation

$$(1) \quad \ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where  $f$  and  $g$  are non-zero polynomials such that  $g(x) = x + g_2(x)$  with  $g_2(0) = 0$  and  $g'_2(0) = 0$ . As usual the dot denotes derivative with respect to time  $t$ . There is no doubt about the importance of the differential equation (1) and this is one of the reasons why it has been studied by so many authors. For instance, if one enters the four words Liénard, polynomial, differential, equation or system into MathSciNet, one would receive 157 articles at the time that this paper is being written.

The differential equation (1) of second order can be written as the following differential system of first order

$$(2) \quad \dot{x} = -y, \quad \dot{y} = x + g_2(x) - f(x)y,$$

where

$$g_2(x) = \sum_{j=2}^{\ell} a_j x^j, \quad f(x) = \sum_{j=0}^m b_j x^j,$$

with  $a_\ell b_m \neq 0$ . Clearly the origin  $(0, 0)$  is a singular point of the differential system (2).

The differential system (2) is called a *generalized polynomial differential Liénard system* or simply *Liénard system*.

The next result characterize the generalized polynomial differential Liénard system (2) having a global center at the origin.

**Theorem 1.** *The generalized polynomial differential Liénard system (2) with  $f(x)$  and  $g(x)$  such that  $g(0) = 0$ ,  $g'(0) > 0$  has a global center at the origin if and only if the following conditions hold:*

- (i) *The unique real root of the polynomial  $g(x)$  is  $x = 0$ .*
- (ii) *There exist real polynomials  $h$ ,  $f_1$  and  $g_1$  such that*

$$f(x) = f_1(h(x))h'(x), \quad g(x) = g_1(h(x))h'(x),$$

*with  $h'(0) = 0$  and  $h''(0) \neq 0$ .*

- (iii)  *$\deg g = \ell$  is odd, and  $\deg g > 1 + \deg f$ .*
- (iv) *The local phase portrait of the singular point localized at the origin of the polynomial differential system*

$$(3) \quad \dot{u} = uv^{\ell-1}f\left(\frac{u}{v}\right) - uv^\ell g\left(\frac{u}{v}\right) - v^{\ell-1}, \quad \dot{v} = v^\ell \left(f\left(\frac{u}{v}\right) - vg\left(\frac{u}{v}\right)\right),$$

*is formed by two hyperbolic sectors.*

Theorem 1 will be proved in section 3.

Now we shall prove that there exist Liénard system (2) satisfying the three conditions of Theorem 1. In other words, we shall prove that there are Liénard system (2) having a global center.

It follows from condition (iii) of Theorem 1 that 3 is the lowest degree of the Liénard systems (2) which can have a global center. The next result classifies all Liénard systems (2) of degree 3 having a global center.

**Theorem 2.** *All Liénard systems (2) of degree 3 having a global center at the origin of coordinates after a rescaling of the variables  $x$ ,  $y$  and  $t$  can be written as*

$$(4) \quad \dot{x} = -y, \quad \dot{y} = x + bx^3 - xy,$$

with  $b > 1/8$ .

Theorem 2 will be proved in section 4.

In the next proposition, we present a Liénard system (2) of degree 5 having a global center.

**Proposition 3.** *The following Liénard system*

$$(5) \quad \dot{x} = -y, \quad \dot{y} = x + 2x^5 - 2xy,$$

*has a global center at the origin of coordinates.*

The proof of Proposition 3 is given in section 5.

## 2. PRELIMINARY RESULTS

We introduce some preliminary results necessary for proving Theorem 3.

The centers are classified in three types. If the Jacobian of the system evaluated at a center has purely imaginary eigenvalues then we say that it is a *linear type center*. If it has both eigenvalues zero but its linear part is not identically zero we call it a *nilpotent type center*. Finally if it has its linear part identically zero then it is a *degenerate center*.

First we recall Theorem 6 of Christopher [2] in which he gave an algebraic and effective characterization in order that a polynomial differential Liénard system (2) has a linear type center at the origin of coordinates.

**Theorem 4.** *The polynomial differential Liénard system (2) with  $f(x)$  and  $g(x)$  real polynomials with  $g(0) = 0$ ,  $g'(0) > 0$  has a linear type center at the origin if and only if there exist real polynomials  $h$ ,  $f_1$  and  $g_1$  such that*

$$f(x) = f_1(h(x))h'(x), \quad g(x) = g_1(h(x))h'(x),$$

with  $h'(0) = 0$  and  $h''(0) \neq 0$ .

We need the notion of *Poincaré disc*. Roughly speaking the *Poincaré disc*  $\mathbb{D}^2$  is the unit closed disc centered at the origin of coordinates whose interior is diffeomorphic with  $\mathbb{R}^2$  and whose boundary, the circle  $S^1$  is identified with the infinity of  $\mathbb{R}^2$ . Note that we can go to the infinity in the plane  $\mathbb{R}^2$  in as many as directions as points has the circle  $S^1$ . Any polynomial differential system can be extended analytically to the whole Poincaré disc, being the circle of the infinity invariant by this extended flow. This extension is called the *Poincaré compactification*, see for details Chapter 5 of [6].

Consider a polynomial differential system in  $\mathbb{R}^2$  whose line at infinity is not filled of singular points and with a unique finite singular point which is a center (of any type). The following result characterizes when this center is global.

**Proposition 5.** *Consider a polynomial differential system in  $\mathbb{R}^2$  whose line at infinity is not filled of singular points and with a unique finite singular point which is a center of a given type. Then this center is global if and only if all the infinite singular points in the Poincaré disc, if they exist, are such that their local phase portraits are formed by two hyperbolic sectors, having their two separatrices on the infinite circle.*

*Proof.* Assume that the polynomial differential system in  $\mathbb{R}^2$  has a unique finite singular point which is a center of a given type and that this center is global. Then the exterior boundary of the period annulus of this center is the circle at infinity. Consequently, since the infinite circle is not filled up with singular points, if there is some infinite singular point this must be formed by two hyperbolic sectors having all of them both separatrices on the infinite circle.

Now assume that the polynomial differential system in  $\mathbb{R}^2$  has a unique finite singular point which is a center of any type, and that all the infinite singular points, if they exist, are such that their local phase portraits are formed by two hyperbolic sectors having all of them both separatrices on the infinite circle. Then consider the period annulus of the center. Its inner boundary is the center, its outer boundary  $\gamma$  is a curve homeomorphic to a circle. If the circle  $\gamma$  is contained in  $\mathbb{R}^2$ , since the unique finite singular point is the center, it must be a periodic orbit, but we claim that this is not possible. Indeed consider a local transversal section  $\Sigma$  to the periodic orbit  $\gamma$  and the Poincaré map  $\pi$  defined on  $\Sigma$ . Then  $\pi$  on the part of  $\Sigma$  contained in the period annulus is the identity. Since  $\pi$  is an analytic function of one variable, because the polynomial differential system is an analytic differential system, it follows  $\pi$  is also the identity on the part of  $\Sigma$  outside the period annulus. So  $\gamma$  is contained in the interior of the period annulus, a contradiction. Hence the claim is proved.

Since the boundary of the period annulus, the circle  $\gamma$  cannot be contained in  $\mathbb{R}^2$ , this boundary must contain some points of the infinite circle, but since

all the infinite singular points, if they exist, its local phase portrait is formed by two hyperbolic sectors having all of them both separatrices on the infinite circle, the boundary  $\gamma$  is the infinite circle. Hence the center is global.  $\square$

In order to be able to apply Proposition 5 we need to study the infinite singular points of system (2). Using the notation of Chapter 5 of [6] we must study the infinite singular points of the local chart  $U_1$  and the origin of the local chart  $U_2$  the remaining infinite singular points are the diametrically opposite singular points in the Poincaré disc. From [6] we know that in order to study the infinite singular points of the extended polynomial differential system

$$(6) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

of degree  $n$  (i.e.  $n$  is the maximum degree of the polynomials  $P$  and  $Q$ ) in the local chart  $U_1$  we must study the singular points of the form  $(u, 0)$  of the polynomial differential system

$$(7) \quad \begin{aligned} \dot{u} &= v^d \left[ -uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \\ \dot{v} &= -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right), \end{aligned}$$

and to study the origin of the local chart  $U_2$  of the polynomial system (6) we must study the origin of the polynomial differential system

$$(8) \quad \begin{aligned} \dot{u} &= v^d \left[ P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \\ \dot{v} &= -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right). \end{aligned}$$

**Proposition 6.** *An infinite singular point  $p$  in the local chart  $U_1$  (respectively  $U_2$ ) whose local phase portrait is formed by two hyperbolic sectors having both separatrices on the infinite circle satisfies that the linear part of system (7) (respectively (8)) at  $p$  is identically zero.*

*Proof.* It is well known that if the linear part of an isolated singular point  $p$  of a differential system in  $\mathbb{R}^2$  has some eigenvalue non-zero, or both eigenvalues zero but the linear part is not identically zero, the local phase portrait of  $p$  cannot be formed by two hyperbolic sectors having both separatrices tangent to the same straight line  $L$  through  $p$ , with one separatrix tangent to one component of  $L \setminus \{p\}$  and the other tangent to the other component of  $L \setminus \{p\}$ . See for details Theorems 2.15, 2.19 and 3.5 of [6].  $\square$

**Theorem 7.** *A polynomial differential system  $\dot{x} = P(x, y)$ ,  $\dot{y} = Q(x, y)$ , of even degree has no global centers.*

Galeotti and Villarini [7] proved that every polynomial differential system of even degree has at least one unbounded orbit. Therefore if a polynomial

differential system has a global center its degree must be odd, this proves Theorem 7. For a different and shorter proof of this result see [11].

### 3. PROOF OF THEOREM 1

If system (2) has a global center at the origin, the origin must be the unique finite singular point, and from (2) this is equivalent to say that  $x = 0$  is the unique real root of the polynomial  $g(x)$ , i.e. the polynomial  $g(x)/x = 1 + g_2(x)/x$  has no real roots. Note that in particular the degree of the polynomial  $g(x)$  must be odd.

From Theorem 7 if a system (2) has a global center we must assume that the degree of such polynomial differential is odd.

In summary, in order that system (2) has a global center we need that  $x = 0$  be the unique real root of the polynomial  $g(x)$  (condition (i)), that the degree  $d = \max\{\ell = 2n - 1, m + 1\}$  of system (2) be odd, and that the origin be a center, i.e. from Theorem 4 that condition (ii) holds.

The polynomial Liénard differential system (2) in the local chart  $U_1$  using (7) becomes

$$(9) \quad \begin{aligned} \dot{u} &= u^2 v^{d-1} + v^{d-1} + \sum_{j=2}^{2n-1} a_j v^{d-j} - u \sum_{j=0}^m b_j v^{d-1-j}, \\ \dot{v} &= uv^d. \end{aligned}$$

We consider three different cases.

*Case 1:*  $2n - 1 > m + 1$ . In this case we have that  $d = 2n - 1$  and  $a_{2n-1} \neq 0$ . So  $\dot{u}|_{v=0} = a_{2n-1} \neq 0$ . Therefore there are no infinite singular points in the local chart  $U_1$ .

*Case 2:*  $2n - 1 = m + 1$ . In this case we have that  $d = 2n - 1 = m + 1$  and  $a_{2n-1}b_m \neq 0$ . So  $\dot{u}|_{v=0} = a_{2n-1} - ub_m$ . Hence there is a unique singular point  $(u^*, 0) = (a_{2n-1}/b_m, 0)$  in the local chart  $U_1$ . The linear part of system (9) at the singular point  $(u^*, 0)$  is

$$\begin{pmatrix} -b_m & a_{2n-2} - \frac{a_{2n-1}}{b_m} b_{m-1} \\ 0 & 0 \end{pmatrix}.$$

Since this linear part is not identically zero, by Propositions 5 and 6 we know that the singular point  $(u^*, 0)$  cannot be formed by two hyperbolic sectors having both separatrices on  $v = 0$ , i.e. on the line of infinity. Therefore in this case the origin of coordinates cannot be a global center.

*Case 3:*  $2n - 1 < m + 1$ . In this case we have that  $d = m + 1$  and  $b_m \neq 0$ . So  $\dot{u}|_{v=0} = -ub_m$ . Hence there is a unique singular point  $(u^*, 0) = (0, 0)$  in

the local chart  $U_1$ . The linear part of system (9) at the singular point  $(0, 0)$  is

$$\begin{pmatrix} -b_m & * \\ 0 & 0 \end{pmatrix}$$

where  $*$  can be zero or not. Since this linear part is not identically zero, again the singular point  $(0, 0)$  cannot be formed by two hyperbolic sectors having the separatrices on the line of infinity. Therefore in this case the origin  $O$  is not a global center.

In short, in order that a polynomial Liénard differential system (2) can have a global center it must be in Case 1, i.e. condition (iii).

The unique infinite singular point under condition (iii) can be the origin of the local chart  $U_2$ . Using (8) the expression of system (2) in the local chart  $U_2$  is the system (4), from it is clear that the origin is a singular point. Then from Proposition 5 the local phase portrait of the origin of coordinates of the local chart  $U_2$  must be formed by two hyperbolic sectors having both separatrices contained in the infinite circle, i.e. condition (iv).

This completes the proof of Theorem 1.

#### 4. PROOF OF THEOREM 2

We are looking for the system (2) satisfying the four conditions of Theorem 1 having the minimum degrees for the polynomials  $f(x)$  and  $g(x)$ .

The minimum degree for the polynomial  $h(x)$  satisfying condition (ii) of Theorem 1 is 2, so for obtaining the minimum degrees for the polynomials  $f(x)$  and  $g(x)$  we must take  $h(x) = cx^2$  with  $c \neq 0$ .

From condition (iii) of Theorem 1 we know that in order that system (2) can have a global center  $\ell > m + 1$  with  $\ell$  odd. The minimum values of  $\ell$  and  $m$  satisfying this relation are  $\ell = 3$  and  $m = 0$ .

From condition (ii) of Theorem 1 we have that  $f(x) = f_1(cx^2)2cx$  and  $g(x) = g_1(cx^2)2cx$ . Since the degree  $m$  of the polynomial  $f(x)$  is zero,  $f(x) = a_1$  with  $a_1 \in \mathbb{R} \setminus \{0\}$  (note that  $a_1$  cannot be zero, otherwise system (2) would not be a polynomial Liénard differential system). Since the degree  $\ell$  of the polynomial  $g(x) = g_1(cx^2)2cx = x + g_2(x)$  is 3, we have that and  $g_1(x) = b_1x + 1/(2c)$  with  $b_1 \neq 0$ . In summary,

$$g(x) = x + b_1c^2x^3 = x + bx^3, \quad f(x) = 2a_1c_1x = ax.$$

Therefore system (2) having the polynomials  $f(x)$  and  $g(x)$  with the minimum degrees satisfying conditions (i), (ii) and (iii) of Theorem 1 are

$$(10) \quad \dot{x} = -y, \quad \dot{y} = x + bx^3 - axy,$$

with  $b > 0$ , by condition (i), and  $a \neq 0$  in order to have a polynomial Liénard differential system.

In order to reduce the number of parameters from two to one, we do the rescaling  $(x, y, t) = (X/a, Y/a)$  and system (10) becomes

$$(11) \quad \dot{x} = -y, \quad \dot{y} = x + bx^3 - xy,$$

with  $b > 0$ , where we have written  $x$  and  $y$  instead of  $X$  and  $Y$ , respectively.

Now we shall study when system (11) satisfies condition (iv) of Theorem 1. So we write system (11) in the local chart  $U_2$  using (11) and we get

$$(12) \quad \dot{u} = -v^2 + u^2v - bu^4 - u^2v^2, \quad \dot{v} = -bu^3v + uv^2 - uv^3.$$

Clearly the origin  $(u, v) = (0, 0)$  is an infinite singular point whose linear part is identically zero, so for studying its local phase portrait we must do the changes of variables called blow ups, see for more details [1].

The characteristic directions are the directions in which the orbits of a differential system can reach or escape from a singular point, see for more details the definition of characteristic orbit in [6]. Since  $u = 0$  is not a characteristic direction because the unique characteristic direction at the origin of system (12) is  $v = 0$ , we start the study of the local phase portrait at the origin of system (12) doing a vertical blow up, i.e. we pass from the variables  $(u, v)$  to the variables

$$(13) \quad (u, v) \rightarrow (u_1, v_1), \quad \text{where } u_1 = u \text{ and } v_1 = v/u.$$

Then system (12) becomes

$$\dot{u}_1 = -u_1^2(bu_1^2 - u_1v_1 + v_1^2 + u_1^2v_1^2), \quad \dot{v}_1 = u_1v_1^3.$$

We rescale the independent variable as follows  $dt_1 = u_1 dt$  in order to eliminate the common factor  $u_1$  between  $\dot{u}_1$  and  $\dot{v}_1$ , and we obtain the system

$$(14) \quad \dot{u}_1 = -u_1(bu_1^2 - u_1v_1 + v_1^2 + u_1^2v_1^2), \quad \dot{v}_1 = v_1^3,$$

where now the dot denotes derivative with respect to the new independent variable  $t_1$ . Again the origin of system (14) is the unique singular point of this system on the straight line  $u_1 = 0$ , and its linear part continues being linearly zero. So we need to do another blow up.

The characteristic directions at the origin of system (14) are  $u_1 = 0$  and  $v_1 = 0$ . Since  $u_1 = 0$  is a characteristic direction if we do a new vertical blow up in general we cannot control the orbits of the local phase portrait at the origin of system (14) which arrive or exit tangent to  $u_1 = 0$ . So before doing a vertical blow we translate the direction  $u_1 = 0$  to the direction  $u_1 = v_1$  doing the change of variables

$$(15) \quad (u_1, v_1) \rightarrow (u_2, v_2), \quad \text{where } u_2 = u_1 + v_1 \text{ and } v_2 = v_1.$$

In this new variables system (14) writes

$$(16) \quad \begin{aligned} \dot{u}_2 = & -bu_2^3 + (1+3b)u_2^2v_2 - 3(1+b)u_2v_2^2 + (3+b)v_2^3 - u_2^3v_2^2 \\ & + 3u_2^2v_2^3 - 3u_2v_2^4 + v_2^5, \\ \dot{v}_2 = & v_2^3, \end{aligned}$$

The origin is the unique singular point of system (21) on the straight line  $u_2 = 0$ , and since its linear part is identically zero for studying its local phase portrait we do the vertical blow up

$$(17) \quad (u_2, v_2) \rightarrow (u_3, v_3), \quad \text{where } u_3 = u_2 \text{ and } v_3 = v_2/u_2,$$

Therefore system (16) becomes

$$\begin{aligned} \dot{u}_3 &= u_3^3(-b + (1 + 3b)v_3 - 3(1 + b)v_3^2 + (3 + b)v_3^3 - u_3^2v_3^2 \\ &\quad + 3u_3^2v_3^3 - 3u_3^2v_3^4 + u_3^2v_3^5), \\ \dot{v}_3 &= -u_3^2(-1 + v_3)v_3(b - (1 + 2b)v_3 + (3 + b)v_3^2 + u_3^2v_3^2 \\ &\quad - 2u_3^2v_3^3 + u_3^2v_3^4), \end{aligned}$$

We rescale the independent variable as follows  $dt_2 = u_3^2 dt_1$  in order to eliminate the common factor  $u_3^2$  between  $\dot{u}_3$  and  $\dot{v}_3$ , and we obtain the system

$$\begin{aligned} \dot{u}_3 &= u_3(-b + (1 + 3b)v_3 - 3(1 + b)v_3^2 + (3 + b)v_3^3 - u_3^2v_3^2 \\ &\quad + 3u_3^2v_3^3 - 3u_3^2v_3^4 + u_3^2v_3^5), \\ (18) \quad \dot{v}_3 &= -(-1 + v_3)v_3(b - (1 + 2b)v_3 + (3 + b)v_3^2 + u_3^2v_3^2 \\ &\quad - 2u_3^2v_3^3 + u_3^2v_3^4), \end{aligned}$$

*Case 1:* Assume  $b > 1/8$ . Then system (18) has the two singular points  $(0, 0)$  and  $(0, 1)$  on the straight line  $u_3 = 0$ . The eigenvalues of the linear part of the system at  $(0, 0)$  are  $\pm b$ , so this point is a saddle (see for instance Theorem 2.15 of [6]). While the eigenvalues of the linear part of the system at  $(0, 1)$  are  $1$  and  $-2$ , so it is also a saddle.

If  $b > 8$  the unique two singular points of system (18) on the straight line  $u_3 = 0$  are the two saddles  $(0, 0)$  and  $(0, 1)$ . Therefore the local phase portrait near the straight line  $u_3 = 0$  for system (18) is topologically equivalent to the one of Figure 1(a).

Now undoing the rescaling  $dt_2 = u_3^2 dt_1$  and going back through the change of variables (17) the phase portrait of Figure 1(a) provides the local phase portrait at the origin of system (16) which is topologically equivalent to the one of Figure 1(b).

Going back through the change of variables (15) the phase portrait of Figure 1(b) provides the local phase portrait at the origin of system (14) which is topologically equivalent to the one of Figure 1(c).

Undoing the rescaling  $dt_1 = u_1 dt$  and going back through the change of variables (13) the phase portrait of Figure 1(c) provides the local phase portrait at the origin of system (12) which is topologically equivalent to the one of Figure 1(d). Hence the local phase portrait at the origin of the local

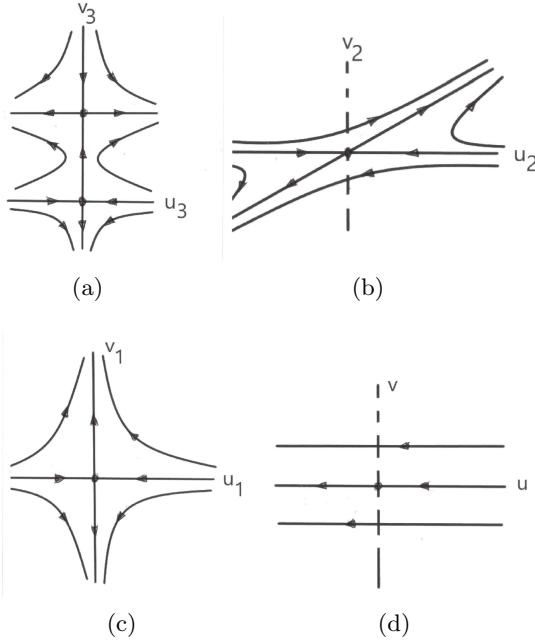


FIGURE 1. The local phase portraits of the blow ups for obtaining the local phase portrait at the origin of the local chart  $U_2$  of system (12).

chart  $U_2$  is formed by two hyperbolic sectors having the two separatrices on the infinite circle.

In summary, from Theorem 1 it follows that Theorem 2 is proved if we show that when  $b \in (0, 1/8]$  the local phase portrait at the origin of system (12) is not formed by two hyperbolic sectors having the two separatrices on the infinite circle.

*Case 2:* Assume  $b = 1/8$ . Then system (18) has the three singular points  $(0, 0)$ ,  $(0, 1/5)$  and  $(0, 1)$  on the straight line  $u_3 = 0$ . As in Case 1 the singular points  $(0, 0)$  and  $(0, 1)$  are saddles. Since the two eigenvalues of the linear part of the system at the singular point  $(0, 1/5)$  are 0 and  $1/25$ , this singular point is semihyperbolic. Using Theorem 2.19 of [6] the local phase portrait at  $(0, 1/5)$  is a saddle-node. Then the parabolic sector providing the nodal part of the saddle-node is not destroyed going back to system (12), consequently the origin of the local chart at  $U_2$  has a parabolic sector, and it cannot be formed by two hyperbolic sectors.

*Case 3: Assume  $b \in (0, 1/8)$ .* Then system (18) has the four singular points  $(0, 0)$ ,  $(0, 1)$  and  $p_{\pm} = (0, (1 + 2b \pm \sqrt{1 - 8b})/(6 + 2b))$  on the straight line  $u_3 = 0$ . As in Cases 1 and 2 the singular points  $(0, 0)$  and  $(0, 1)$  are saddles.

The determinat of the linear part of the system at the singular points  $p_{\pm}$  is

$$\det_{\pm} = \frac{(8b-1)(b((b-12)b-4)-2)}{2(b+3)^4} \pm \frac{\sqrt{1-8b}(b(b(17b-20)-4)+2)}{2(b+3)^4}.$$

Since  $\det_+ \det_- = 2b^3(8b-1)/(b+3)^4 < 0$ , so we have a saddle and a node. As in Case 2 the parabolic sector of the node persist going back to system (12), and the origin of the local chart at  $U_2$  has a parabolic sector, and it cannot be formed by two hyperbolic sectors.

In summary the proof of Theorem 2 is complete.

## 5. PROOF OF PROPOSTION 3

First we shall prove that the polynomial Liénard differential system (5) has a unique finite singular point, the origin of coordinates, which is a center. Indeed, it is easy to check that the origin is the unique finite singular of system (5). Now note that if  $h(x) = x^2$ ,  $f_1(x) = 1$  and  $g_1(x) = x^2 + 1/2$ , then

$$f(x) = f_1(h(x))h'(x) = 2x, \quad \text{and} \quad g(x) = g_1(h(x))h'(x) = x + 2x^5.$$

So, since  $g(0) = 0$ ,  $g'(0) = 1 > 0$ ,  $h'(0) = 0$ ,  $h''(0) = 2 \neq 0$ , by Theorem 2 the Liénard differential system (5) has a center at the origin.

Since we are in the previous Case 1 because using the notation of that case we have  $2n-1 = 5$  and  $m+1 = 1$ , system (5) has no infinite singular points in the local chart  $U_1$ . Then, from the previous arguments done before starting the proof of Theorem 3, in order to complete the proof of this theorem we only need to show that the origin of the local chart  $U_2$  is a singular point such that its local phase portrait is formed by two hyperbolic sectors having their separatrices contained in the infinite circle. Indeed, from (8) system (5) in the local chart  $U_2$  writes

$$(19) \quad \begin{aligned} \dot{u} &= -v^4 + 2u^2v^3 - 2u^6 - u^2v^4, \\ \dot{v} &= 2uv^4 - 2u^5v - uv^5. \end{aligned}$$

Clearly the origin of system (19) is an infinite singular point whose linear part is identically zero, so for studying its local phase portrait we must do the changes of variables called blow ups.

Since  $u = 0$  is not a characteristic direction because the unique characteristic direction at the origin of system (19) is  $v = 0$ , we start the study of the local phase portrait at the origin of system (19) doing a vertical blow up, i.e. we pass from the variables  $(u, v)$  to the variables  $(u_1, v_1)$  using (13). Then system (19) becomes

$$\begin{aligned} \dot{u}_1 &= -u_1^4(2u_1^2 - 2u_1v_1^3 + v_1^4 + u_1^2v_1^4), \\ \dot{v}_1 &= u_1^3v_1^5. \end{aligned}$$

We rescale the independent variable as follows  $dt_1 = u_1^3 dt$  in order to eliminate the common factor  $u_1^3$  between  $\dot{u}_1$  and  $\dot{v}_1$ , and we obtain the system

$$(20) \quad \begin{aligned} \dot{u}_1 &= -u_1(2u_1^2 - 2u_1v_1^3 + v_1^4 + u_1^2v_1^4), \\ \dot{v}_1 &= v_1^5, \end{aligned}$$

where now the dot denotes derivative with respect to the new independent variable  $t_1$ . Again the origin of system (20) is the unique singular point of this system on the straight line  $u_1 = 0$ , and its linear part continues being linearly zero. So we need to do another blow up.

The characteristic directions at the origin of system (20) are  $u_1 = 0$  with multiplicity three and  $v_1 = 0$ . Since  $u_1 = 0$  is a characteristic direction we translate the direction  $u_1 = 0$  to the direction  $u_1 = v_1$  doing the change of variables (15). In this new variables system (20) writes

$$(21) \quad \begin{aligned} \dot{u}_2 &= -2u_2^3 + 6u_2^2v_2 - 6u_2v_2^2 + 2v_2^3 + 2u_2^2v_2^3 - 5u_2v_2^4 + 4v_2^5 \\ &\quad -u_2^3v_2^4 + 3u_2^2v_2^5 - 3u_2v_2^6 + v_2^7, \\ \dot{v}_2 &= v_2^5, \end{aligned}$$

The origin is the unique singular point of system (21) on the straight line  $u_2 = 0$ , and since its linear part is identically zero for studying its local phase portrait we do the vertical blow up (17). Therefore system (21) becomes

$$\begin{aligned} \dot{u}_3 &= u_3^3(-2 + 6v_3 - 6v_3^2 + 2v_3^3 + 2u_3^2v_3^3 - 5u_3^2v_3^4 + 4u_3^2v_3^5 \\ &\quad -u_3^4v_3^4 + 3u_3^4v_3^5 - 3u_3^4v_3^6 + u_3^4v_3^7), \\ \dot{v}_3 &= u_3^2v_3(1 - v_3)(2 - 4v_3 + 2v_3^2 - 2u_3^2v_3^3 + 4u_3^2v_3^4 + u_3^4v_3^4 \\ &\quad -u_3^4v_3^4 - 2u_3^4v_3^5 + u_3^4v_3^6), \end{aligned}$$

We rescale the independent variable as follows  $dt_2 = u_3^2 dt_1$  in order to eliminate the common factor  $u_3^2$  between  $\dot{u}_3$  and  $\dot{v}_3$ , and we obtain the system

$$(22) \quad \begin{aligned} \dot{u}_3 &= u_3(-2 + 6v_3 - 6v_3^2 + 2v_3^3 + 2u_3^2v_3^3 - 5u_3^2v_3^4 + 4u_3^2v_3^5 \\ &\quad -u_3^4v_3^4 + 3u_3^4v_3^5 - 3u_3^4v_3^6 + u_3^4v_3^7), \\ \dot{v}_3 &= v_3(1 - v_3)(2 - 4v_3 + 2v_3^2 - 2u_3^2v_3^3 + 4u_3^2v_3^4 + u_3^4v_3^4 \\ &\quad -u_3^4v_3^4 - 2u_3^4v_3^5 + u_3^4v_3^6), \end{aligned}$$

This system has two singular points on the straight line  $u_3 = 0$ , namely the origin  $(0, 0)$  and  $(0, 1)$ . The eigenvalues of the linear part of the system at  $(0, 0)$  are  $\pm 2$ , so this point is a saddle (see for instance Theorem 2.15 of [6]). The linear part of the system at  $(0, 1)$  is identically zero. Hence in order to know the local phase portrait at the singular point  $(0, 1)$  we must do blow ups.

First we translate the singular point  $(0, 1)$  at the origin of coordinates doing the change of variables

$$(23) \quad (u_3, v_3) \rightarrow (u_4, v_4), \text{ where } u_4 = u_3 \text{ and } v_4 = v_3 - 1.$$

In the new variables system (22) becomes

$$(24) \quad \begin{aligned} \dot{u}_4 &= u_4(u_4^2 + 6u_4^2v_4 + 2v_4^3 + 16u_4^2v_4^2 + 22u_4^2v_4^3 + 15u_4^2v_4^4 + u_4^4v_4^3 \\ &\quad + 4u_4^2v_4^5 + 4u_4^4v_4^4 + 6u_4^4v_4^5 + 4u_4^4v_4^6 + u_4^4v_4^7), \\ \dot{v}_4 &= -v_4(1 + v_4)(2u_4^2 + 2v_4^2 + 10u_4^2v_4 + 18u_4^2v_4^2 + 14u_4^2v_4^3 + u_4^4v_4^2 \\ &\quad + 4u_4^2v_4^4 + 4u_4^4v_4^3 + 6u_4^4v_4^4 + 4u_4^4v_4^5 + u_4^4v_4^6). \end{aligned}$$

This system has the two singular points  $(0, 0)$  and  $(0, -1)$ . Since the linear part of the system at  $(0, 0)$  is identically zero, we must study its local phase portraits doing blow ups, but due to the fact that the origin has the two characteristic directions  $u_4 = 0$  and  $v_4 = 0$  we cannot do a vertical blow up to system (24) because we can lost information around the straight line  $u_4 = 0$ . So we pass the straight line  $u_4 = 0$  to the straight line  $u_4 = v_4$  doing the change of variables

$$(25) \quad (u_4, v_4) \rightarrow (u_5, v_5), \quad \text{where } u_5 = u_4 + v_4 \text{ and } v_5 = v_4.$$

Hence system (24) in the new variables writes

$$(26) \quad \begin{aligned} \dot{u}_5 &= u_5^3 - 5u_5^2v_5 + 7u_5v_5^2 - 5v_5^3 + 6u_5^3v_5 - 30u_5^2v_5^2 + 44u_5v_5^3 - 22v_5^4 \\ &\quad + 16u_5^3v_5^2 - 76u_5^2v_5^3 + 104u_5v_5^4 - 44v_5^5 + 22u_5^3v_5^3 - 98u_5^2v_5^4 \\ &\quad + 130u_5v_5^5 - 54v_5^6 - u_5^4v_5^3 + 19u_5^3v_5^4 - 69u_5^2v_5^5 + 85u_5v_5^6 - 34v_5^7 \\ &\quad + u_5^5v_5^3 - 10u_5^4v_5^4 + 34u_5^3v_5^5 - 56u_5^2v_5^6 + 45u_5v_5^7 - 14v_5^8 + 4u_5^5v_5^4 \\ &\quad - 30u_5^4v_5^5 + 80u_5^3v_5^6 - 100u_5^2v_5^7 + 60u_5v_5^8 - 14v_5^9 + 6u_5^5v_5^5 \\ &\quad - 40u_5^4v_5^6 + 100u_5^3v_5^7 - 120u_5^2v_5^8 + 70u_5v_5^9 - 16v_5^{10} + 4u_5^5v_5^6 \\ &\quad - 25u_5^4v_5^7 + 60u_5^3v_5^8 - 70u_5^2v_5^9 + 40u_5v_5^{10} - 9v_5^{11} + u_5^5v_5^7 - 6u_5^4v_5^8 \\ &\quad + 14u_5^3v_5^9 - 16u_5^2v_5^{10} + 9u_5v_5^{11} - 2v_5^{12}, \\ \dot{v}_5 &= -v_5(1 + v_5)(2u_5^2 - 4u_5v_5 + 4v_5^2 + 10u_5^2v_5 - 20u_5v_5^2 + 10v_5^3 \\ &\quad + 18u_5^2v_5^2 - 36u_5v_5^3 + 18v_5^4 + 14u_5^2v_5^3 - 28u_5v_5^4 + 14v_5^5 + u_5^4v_5^2 \\ &\quad - 4u_5^3v_5^3 + 10u_5^2v_5^4 - 12u_5v_5^5 + 5v_5^6 + 4u_5^4v_5^3 - 16u_5^3v_5^4 + 24u_5^2v_5^5 \\ &\quad - 16u_5v_5^6 + 4v_5^7 + 6u_5^4v_5^4 - 24u_5^3v_5^5 + 36u_5^2v_5^6 - 24u_5v_5^7 + 6v_5^8 \\ &\quad + 4u_5^4v_5^5 - 16u_5^3v_5^6 + 24u_5^2v_5^7 - 16u_5v_5^8 + 4v_5^9 + u_5^4v_5^6 - 4u_5^3v_5^7 \\ &\quad + 6u_5^2v_5^8 - 4u_5v_5^9 + v_5^{10}). \end{aligned}$$

Now we do the blow up

$$(27) \quad (u_5, v_5) \rightarrow (u_6, v_6), \quad \text{where } u_6 = u_5 \text{ and } v_6 = v_5/u_5,$$

and system (26) becomes

$$\begin{aligned}
\dot{u}_6 = & -u_6(-1 + 5v_6 - 6u_6v_6 - 7v_6^2 + 30u_6v_6^2 + 5v_6^3 - 16u_6^2v_6^2 \\
& - 44u_6v_6^3 + 76u_6^2v_6^3 + 22u_6v_6^4 - 22u_6^3v_6^3 - 104u_6^2v_6^4 + u_6^4v_6^3 \\
& + 98u_6^3v_6^4 + 44u_6^2v_6^5 - u_6^5v_6^3 - 19u_6^4v_6^4 - 130u_6^3v_6^5 + 10u_6^5v_6^4 \\
& + 69u_6^4v_6^5 + 54u_6^3v_6^6 - 4u_6^6v_6^4 - 34u_6^5v_6^5 - 85u_6^4v_6^6 + 30u_6^6v_6^5 \\
(28) \quad & + 56u_6^5v_6^6 + 34u_6^4v_6^7 - 6u_6^7v_6^5 - 80u_6^6v_6^6 - 45u_6^5v_6^7 + 40u_6^7v_6^6 \\
& + 100u_6^6v_6^7 + 14u_6^5v_6^8 - 4u_6^8v_6^6 - 100u_6^7v_6^7 - 60u_6^6v_6^8 + 25u_6^8v_6^7 \\
& + 120u_6^7v_6^8 + 14u_6^6v_6^9 - u_6^9v_6^7 - 60u_6^8v_6^8 - 70u_6^7v_6^9 + 6u_6^9v_6^8 \\
& + 70u_6^8v_6^9 + 16u_6^7v_6^{10} - 14u_6^9v_6^9 - 40u_6^8v_6^{10} + 16u_6^9v_6^{10} + 9u_6^8v_6^{11} \\
& - 9u_6^9v_6^{11} + 2u_6^9v_6^{12}), \\
\dot{v}_6 = & (v_6 - 1)v_6(3 - 6v_6 + 18u_6v_6 + 5v_6^2 - 36u_6v_6^2 + 44u_6^2v_6^2 + 22u_6v_6^3 \\
& - 88u_6^2v_6^3 + u_6^4v_6^2 + 54u_6^3v_6^3 + 44u_6^2v_6^4 - 4u_6^4v_6^3 - 108u_6^3v_6^4 + 6u_6^5v_6^3 \\
& + 39u_6^4v_6^4 + 54u_6^3v_6^5 - 24u_6^5v_6^4 - 70u_6^4v_6^5 + 14u_6^6v_6^4 + 44u_6^5v_6^5 \\
& + 34u_6^4v_6^6 - 56u_6^6v_6^5 - 40u_6^5v_6^6 + 16u_6^7v_6^5 + 84u_6^6v_6^6 + 14u_6^5v_6^7 \\
& - 64u_6^7v_6^6 - 56u_6^6v_6^7 + 9u_6^8v_6^6 + 96u_6^7v_6^7 + 14u_6^6v_6^8 - 36u_6^8v_6^7 \\
& - 64u_6^7v_6^8 + 2u_6^9v_6^7 + 54u_6^8v_6^8 + 16u_6^7v_6^9 - 8u_6^9v_6^8 - 36u_6^8v_6^9 \\
& + 12u_6^9v_6^9 + 9u_6^8v_6^{10} - 8u_6^9v_6^{10} + 2u_6^9v_6^{11}).
\end{aligned}$$

The unique two singular points of system (5) on the straight line  $u_6 = 0$  are  $(0, 0)$  and  $(0, 1)$ . The eigenvalues of the linear part of the system at  $(0, 0)$  are 1 and  $-3$ , so it is a saddle and the eigenvalues of the linear part of the system at  $(0, 1)$  are  $\pm 2$ , so it is another saddle. Therefore the local phase portrait near the straight line  $u_6 = 0$  for system (5) is topologically equivalent to the one of Figure 2(a).

Going back through the change of variables (27) the phase portrait of Figure 2(a) provides the local phase portrait at the origin of system (26) which is topologically equivalent to the one of Figure 2(b).

Going back through the change of variables (25) from the phase portrait of Figure 2(b) we obtain the local phase portrait at the origin of system (24) which is topologically equivalent to the one of Figure 2(c).

Again going back through the change of variables (23) the phase portrait of Figure 2(c) provides the local phase portrait around the straight line  $u_3 = 0$  of system (22) which is topologically equivalent to the one of Figure 2(d).

Now undoing the rescaling  $dt_2 = u_3^2 dt_1$  and going back through the change of variables (17) the phase portrait of Figure 2(d) provides the local phase

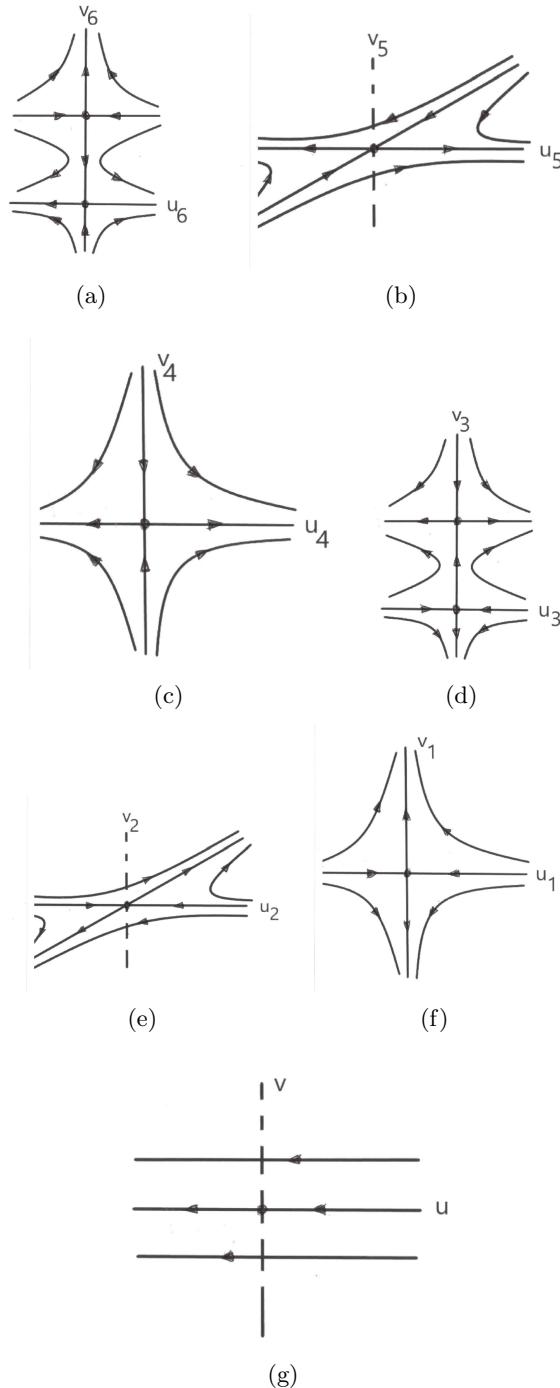


FIGURE 2. The local phase portraits of the blow ups for obtaining the local phase portrait at the origin of the local chart  $U_2$  of system (19).

portrait at the origin of system (21) which is topologically equivalent to the one of Figure 2(e).

Going back through the change of variables (15) the phase portrait of Figure 2(e) provides the local phase portrait at the origin of system (20) which is topologically equivalent to the one of Figure 2(f).

Finally undoing the rescaling  $dt_1 = u_1^3 dt$  and going back through the change of variables (13) the phase portrait of Figure 2(f) provides the local phase portrait at the origin of system (19) which is topologically equivalent to the one of Figure 2(g). Hence the origin of the local chart  $U_2$  has a local phase portrait formed by two hyperbolic sectors having their two separatrices on the infinite circle. This completes the proof of Proposition 3

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