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RATIONAL LIMIT CYCLES OF ABEL EQUATIONS

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Abstract. We deal with Abel equations $dy/dx = A(x)y^2 + B(x)y^3$, where $A(x)$ and $B(x)$ are real polynomials. We prove that these Abel equations can have at most two rational limit cycles and we characterize when this happens. Moreover we provide examples of these Abel equations with two nontrivial rational limit cycles.

1. Introduction and statement of the results.

We study the Abel equations

$$\frac{dy}{dx} = A(x)y^2 + B(x)y^3, \tag{1}$$

where $x \in [0,1]$ and y are real variables and $A(x)$ and $B(x)$ are polynomials. The limit cycles of these equations have been intensively investigated mainly when the functions $A(x)$ and $B(x)$ are periodic (see for instance [1, 2, 3, 4, 5, 6, 9, 11, 14, 15, 17, 18, 19, 20, 21, 23, 24, 25, 26]), and also when $A(x)$ and $B(x)$ are polynomial (see for instance [10, 12, 13, 16, 22]). Here we are interested in the rational limit cycles of equation (1) when the functions $A(x)$ and $B(x)$ are polynomials.

A *periodic solution* of equation (1) is a solution $y(x)$ defined in the closed interval $[0,1]$ such that $y(0) = y(1)$, note that without loss of generality we are assuming that the period is 1. We say that a *limit cycle* is a periodic solution isolated in the set of periodic solutions of a differential equation (1).

The limit cycle is called a *polynomial limit cycle* if the periodic solution $y(x)$ is a polynomial in the variable x. In particular the authors of [16] proved that any polynomial limit cycle of the differential equation (1) is of the form $y = c$ with $c \in \mathbb{R}$, and that if a polynomial limit cycle exists with $c \neq 0$, then no other polynomial limit cycles can exist.

In this paper we study the existence of *rational limit cycles* for the differential equation (1), i.e. we want to consider limit cycles of the form $y(x) = q(x)/p(x)$

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