# DYNAMICS OF THE SZEKERES SYSTEM

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Abstract. The Szekeres model is a four-dimensional system which are the exact solutions of the Einstein field equations when there exists irrotational dust. It is completely integrable with two rational and one analytic first integral. We describe the dynamics of the Szekeres system for any of the values of these two rational first integrals.

## 1. Introduction and statement of the results

The exact solutions of the Einstein field equations with irrational dust are modelized by the Szekeres model (a four-dimensional system introduced in [15]). We note that the Lemaître-Tolman models (see [2] for details) can be obtained by the limiting cases of the Szekeres models.

The equations of motion of the Szekeres system are

$$\begin{aligned}
\dot{\rho} &= -\theta\rho,\\
\dot{\theta} &= -\frac{1}{3}\theta^2 - 6\sigma^2 - \frac{1}{2}\rho,\\
\dot{\sigma} &= \sigma^2 - \frac{2}{3}\theta\sigma - E,\\
\dot{E} &= -3E\sigma - \theta E - \frac{1}{2}\rho\sigma,
\end{aligned} \tag{1}$$

where $\rho$ is the energy density, $\theta$ is the expansion scalar, $\sigma$ is the shear and $E$ is the Weyl tensor and the dot means derivative with respect to $t$.

As it is shown in [3], the Silent Universe system with $\sigma_1 = \sigma_2 = \sigma$ and $E_1 = E_2 = E$ yields the Szekeres system (here $\sigma_1$ and $\sigma_2$ are the independent eigenvalues of the traceless shear tensor and $E_1$ and $E_2$ are the traceless components of the Weyl tensor).

In order to describe the propagation of the light in nonhomogeneous universe models, or to analyze the evolution and formation of the structure of the Universe some authors used the Szekeres system, see for instance [1, 8, 9, 14, 16] and the references therein. This clearly justifies the necessity of knowing the dynamics of such system near infinity.

As it was shown in [5] the Szekeres system is completely integrable because it has three functional independent first integrals (two of them being rational and the

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third one being analytic). The two rational first integrals are

$$F = \frac{(-18E - 3\rho + (\theta + 3\sigma)^2)^3}{(6E+\rho)^2},$$
$$H = \frac{(3\theta\sigma(2E+\rho) - E(18E + 2\theta^2 + 3\rho) + 9\sigma^2(4E+\rho))^3}{\rho^3(\rho+6E)^2}.$$

Setting $F = f$ and $H = h$ we will describe the dynamics of the Szekeres system (1) restricted to the invariant set defined by $F = f$ and $H = h$.

The next equality follows from system (1)

$$\theta = -\frac{\dot\rho}{\rho}, \quad \sigma = \frac{2(-\rho\dot E + E\dot\rho)}{\rho(\rho+6E)}.$$

Introducing $\theta$ and $\sigma$ in the equations of $\dot\theta$ and $\dot\sigma$ in (1), solving them with respect to $\ddot\rho$ and $\ddot E$ we get a differential system that has the form $\ddot\rho = f(\rho, E)$, $\ddot E = g(\rho, E)$. Applying to that system the change of variables

$$\rho = \frac{6}{(1-x)y^3}, \quad E = -\frac{x}{(1-x)y^3},$$

we obtain the differential system (1) rewritten as

$$\ddot x + \frac{2\dot x\dot y}{y} - \frac{3x}{y^3} = 0, \quad \ddot y + \frac{1}{y^2} = 0. \tag{2}$$

Such a reduction (going from system (1) to system (2)) was done in [11].

We rewrite system (2) as

$$\begin{aligned} \dot x &= z,\\ \dot y &= w,\\ \dot z &= -\frac{2zw}{y} + \frac{3x}{y^3},\\ \dot w &= -\frac{1}{y^2}. \end{aligned} \tag{3}$$

In these new variables the first integrals $F$ and $H$ of system (3) become

$$\mathcal{F} = w^2 - \frac{2}{y} \quad \text{and} \quad \mathcal{H} = xw^2 + \frac{x}{y} + wyz.$$

Setting $\mathcal{F} = f$ and $\mathcal{H} = h$ we obtain

$$y = \frac{2}{w^2 - f} \quad \text{and} \quad x = -\frac{2(fh + 2wz - hw^2)}{(f-w^2)(f-3w^2)}.$$

Note that here the expressions for $y$ and $x$ are not well defined when $(f - w^2)(f - 3w^2) = 0$. So, if $f = 0$ they are not well defined for $w = 0$, and if $f > 0$ they are not well-defined for $w = \pm\sqrt{f}$ and $w = \pm\sqrt{f/3}$. So in these cases the variables $x$ and $y$ are not defined and so they correspond to *singularities* of the Skezeres system.

The differential system (3) restricted to $\mathcal{F} = f$ and $\mathcal{H} = h$ can be written as

$$\begin{aligned} z' &= \frac{(f-w^2)}{4(f-3w^2)}(3f^2h + 10fzw - 6fhw^2 - 18zw^3 + 3hw^4), \\ w' &= -\frac{1}{4}(f-w^2)^2. \end{aligned} \tag{4}$$

We do a reparametrization of time in the form $dt = (f-3w^2)ds/(f-w^2)$, and we rewrite system (4) as

$$\begin{aligned} \dot{z} &= 3f^2h + 10fzw - 6fhw^2 - 18zw^3 + 3hw^4, \\ \dot{w} &= -\frac{1}{4}(f-w^2)(f-3w^2), \end{aligned} \tag{5}$$

where for this new system the dot means derivative with respect to $s$.

The objective now is to know the dynamics of this differential system for all the values of $f$ and $h$. To do so and taking into account that system (5) is a polynomial differential system, we shall use the Poincaré compactification (see [4, Chapter 5] for details and section 3).

Roughly speaking the $\alpha$-limit of an orbit is the place where it borns and the $\omega$-limit of an orbit is the place where it dies. A more precise definition will be given at the end of section 3. We will investigate the $\alpha$-limit and $\omega$-limit of all the solutions of system (1).

The dynamics of the Szekeres system (1) when $f < 0$ was completely determined in [6], there it is proved the following result.

**Theorem 1.** *All solutions of the Szekeres system* (1) *with $f < 0$ have $\alpha$-limit and $\omega$-limit at infinity, more precisely at the endpoints of the $z$-axis and $w$-axis.*

The main objective of this paper is to study the dynamics of Szekeres system (1) in the remaining cases, that is when either $f = 0$, or $f > 0$.

Our main results are the following two results.

**Theorem 2.** *The following holds for the Szekeres system* (1) *with $f > 0$.*

(a) *In the half-space $w > \sqrt{f}$ all solutions have $\alpha$-limit the infinity at the endpoints of the $z$-axis, and $\omega$-limit the point $(0, \sqrt{f})$ without reaching it (because on that point the dynamics is not defined), except the ones which have $\alpha$-limit an infinite singular point between the endpoints of the $z$-axis and $\omega$-limit the point $(0, \sqrt{f})$.*

(b) *The orbits between the hyperplanes $w = \sqrt{f}$ and $w = \sqrt{f/3}$ have $\alpha$-limit the point $(0, \sqrt{f})$ and $\omega$-limit the point $\big(-\sqrt{\frac{f}{3}}\,h, \sqrt{\frac{f}{3}}\big)$ without reaching them (on these points the dynamics is not defined).*

(c) *The orbits between the hyperplanes $w = \sqrt{f/3}$ and $w = -\sqrt{f/3}$ have $\alpha$-limit the point $\big(-\sqrt{\frac{f}{3}}\,h, \sqrt{\frac{f}{3}}\big)$ and $\omega$-limit the point $\big(\sqrt{\frac{f}{3}}\,h, -\sqrt{\frac{f}{3}}\big)$ without reaching them (on these points the dynamics is not defined).*

(d) *The orbits between the hyperplanes* $w = -\sqrt{f/3}$ *and* $w = -\sqrt{f}$ *have* $\alpha$*-limit the point* $\big(\sqrt{\frac{f}{3}}\,h, -\sqrt{\frac{f}{3}}\big)$ *and* $\omega$*-limit the point* $\big(0, -\sqrt{f}\big)$ *without reaching them (on these points the dynamics is not defined).*

(e) *In the half-space* $w < -\sqrt{f}$ *all solutions have* $\omega$*-limit the infinity at the endpoints of the* $z$*-axis and* $\alpha$*-limit the point* $(0, -\sqrt{f})$ *without reaching it (because on that point the dynamics is not defined), except the ones which have* $\omega$*-limit an infinite singular point between the endpoints of the* $z$*-axis and* $\alpha$*-limit the point* $(0, -\sqrt{f})$*.*

**Theorem 3.** *All solutions of the Szekeres system* (1) *with* $f = 0$ *have* $\alpha$*-limit the infinity at the endpoints of the* $z$*-axis, and* $\omega$*-limit the origin, without crossing the hyperplane* $w = 0$ *where the dynamics is not defined, except the ones which are in* $w > 0$ *and have* $\alpha$*-limit an infinite singular point between the endpoints of the* $z$*-axis and* $\omega$*-limit the origin, and the ones which are in* $w < 0$ *and have* $\omega$*-limit an infinite singular point between the endpoints of the* $z$*-axis and* $\alpha$*-limit the origin.*

Theorems 3 and 2 are proved in section 3.

## 2. Preliminaries

2.1. **Poincaré compactification.** In this section we summarize some basic results about the Poincaré compactification, which was done by Poincaré in [13]. He provided a tool for studying the behaviour of a planar polynomial differential system near the infinity. For more details on the Poincaré compactification, see [4, Chapter 5].

Let $\mathcal{X} = P\dfrac{\partial}{\partial x_1} + Q\dfrac{\partial}{\partial x_2}$ be a polynomial vector field of degree $d$. We consider the *Poincaré sphere* $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$, its tangent plane to the point $(0, 0, 1)$ is identified with $\mathbb{R}^2$. Now we consider the central projection $f : \mathbb{R}^2 \to \mathbb{S}^2$ of the vector field $\mathcal{X}$, which sends every point $x \in \mathbb{R}^2$ to the two intersection points of the straight line passing through the point $x$ and the origin of coordinates with the sphere $\mathbb{S}^2$. We note that the equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ of the sphere is in bijection with the infinity of $\mathbb{R}^2$. The differential $Df$ sends the vector field $\mathcal{X}$ on $\mathbb{R}^2$ into a vector field $\mathcal{X}'$ defined on $\mathbb{S}^2 \setminus \mathbb{S}^1$, which is formed by two symmetric copies of $\mathcal{X}$ with respect to the origin of coordinates.

We can extend the vector field $\mathcal{X}'$ analytically to a vector field on $\mathbb{S}^2$ multiplying $\mathcal{X}'$ by $y_3^d$. This new vector field is denoted by $p(\mathcal{X})$ and it is called the *Poincaré compactification* of the polynomial vector field $\mathcal{X}$ on $\mathbb{R}^2$. The dynamics of $p(\mathcal{X})$ near $\mathbb{S}^1$ corresponds with the dynamics of $\mathcal{X}$ in the neighborhood of the infinity. Since $\mathbb{S}^2$ is a curved surface, for working with the vector field $p(\mathcal{X})$ on $\mathbb{S}^2$, we need the expressions of this vector field in the local charts $(U_i, \phi_i)$ and $(V_i, \psi_i)$, where $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$, $\phi_i : U_i \longrightarrow \mathbb{R}^2$ and $\psi_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1, 2, 3$, with $\phi_i(y) = -\psi_i(y) = (y_m/y_i, y_n/y_i)$ for $m < n$ and $m, n \neq i$. In the local chart $(U_1, \phi_1)$ the expression of $p(\mathcal{X})$ is

$$\dot{u} = v^d \left[ -uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \qquad \dot{v} = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right).$$

In $(U_2,\phi_2)$ the expression of $p(\mathcal{X})$ is

$$\dot{u} = v^d\left[P\left(\frac{u}{v},\frac{1}{v}\right) - uQ\left(\frac{u}{v},\frac{1}{v}\right)\right], \qquad \dot{v} = -v^{d+1}Q\left(\frac{u}{v},\frac{1}{v}\right),$$

and for $(U_3,\phi_3)$ is

$$\dot{u} = P(u,v), \qquad \dot{v} = Q(u,v).$$

The expressions for $p(\mathcal{X})$ in the local chart $(V_i,\psi_i)$ is the same as in the local chart $(U_i,\phi_i)$ multiplied by $(-1)^{d-1}$ for $i=1,2,3$. The points of $\mathbb{S}^1$ in any local chart have its $v$ coordinate equal to zero.

We note that the equator $\mathbb{S}^1$ is invariant by the vector field $p(\mathcal{X})$. The *infinite singular points* of $\mathcal{X}$ are the singular points of $p(\mathcal{X})$ which lie in $\mathbb{S}^1$. Note that if $y \in \mathbb{S}^1$ is an infinite singular point, then $-y$ is also an infinite singular point and these two points have the same stability if the degree of vector field is odd. Such stability change to the opposite if the degree of the vector field is even.

The image of the northern hemisphere of $\mathbb{S}^2$ onto the plane $y_3 = 0$ under the projection $\pi(y_1,y_2,y_3) = (y_1,y_2)$ is called the *Poincaré disc* which is denoted by $\mathbb{D}$. The integral curves of $\mathbb{S}^2$ are symmetric with respect to the origin, therefore it is sufficient to investigate the flow of $p(\mathcal{X})$ only in the closed northern hemisphere. In order to draw the phase portrait on the Poincaré disc it is needed to project by $\pi$ the phase portrait of $p(\mathcal{X})$ on the northern hemisphere of $\mathbb{S}^2$.

We note that the points $(u,0)$ are the points at infinity in the local charts $U_i$ and $V_i$ with $i=1,2$. Moreover, we remark that for studying the infinite singularities it is sufficient to study them on the local chart $U_1$, and to check if the origin of the local chart $U_2$ is or not a singularity.

2.2. **Topological equivalence of two polynomial vector fields.** Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two polynomial vector fields on $\mathbb{R}^2$. We say that they are *topologically equivalent* if there exists a homeomorphism on the Poincaré disc $\mathbb{D}$ which preserves the infinity $\mathbb{S}^1$ and sends the orbits of $\pi(p(\mathcal{X}_1))$ to orbits of $\pi(p(\mathcal{X}_2))$, preserving or reversing the orientation of all the orbits.

A *separatrix* of the Poincaré compactification $\pi(p(\mathcal{X}))$ is one of following orbits: all the orbits at the infinity $\mathbb{S}^1$, the finite singular points, the limit cycles, and the two orbits at the boundary of a hyperbolic sector at a finite or an infinite singular point, see for more details on the separatrices [7, 10].

The set of all separatrices of $\pi(p(\mathcal{X}))$, which we denote by $\Sigma_{\mathcal{X}}$, is a closed set (see [10]).

A *canonical region* of $\pi(p(\mathcal{X}))$ is an open connected component of $\mathbb{D} \setminus \Sigma_{\mathcal{X}}$. The union of the set $\Sigma_{\mathcal{X}}$ with an orbit of each canonical region form the *separatrix configuration* of $\pi(p(\mathcal{X}))$ and is denoted by $\Sigma'_{\mathcal{X}}$. We denote the number of separatrices of a phase portrait in the Poincaré disc by $S$, and its number of canonical regions by $R$.

Two separatrix configurations $\Sigma'_{\mathcal{X}_1}$ and $\Sigma'_{\mathcal{X}_2}$ are *topologically equivalent* if there is a homeomorphism $h : \mathbb{D} \longrightarrow \mathbb{D}$ such that $h(\Sigma'_{\mathcal{X}_1}) = \Sigma'_{\mathcal{X}_2}$.

According to the following theorem which was proved by Markus [7], Neumann [10] and Peixoto [12], it is sufficient to investigate the separatrix configuration of a polynomial differential system, for determining its global phase portrait.

**Theorem 4.** *Two Poincaré compactified polynomial vector fields* $\pi(p(\mathcal{X}_1))$ *and* $\pi(p(\mathcal{X}_2))$ *with finitely many separatrices are topologically equivalent if and only if their separatrix configurations* $\Sigma'_{\mathcal{X}_1}$ *and* $\Sigma'_{\mathcal{X}_2}$ *are topologically equivalent.*

## 3. Proofs

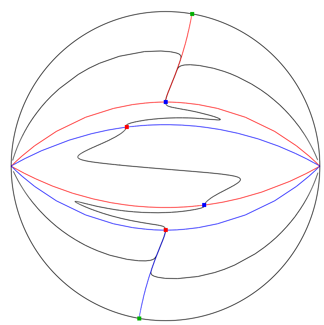

Figure 1. The phase portrait of system (5) in the Poincaré disc when $f > 0$. On the four parallel straight lines $w$ equal to $\pm\sqrt{f}$ and $\pm\sqrt{f/3}$ of this phase portrait of the Szekeres system restricted to $F = f > 0$ and $H = h$ is not defined.

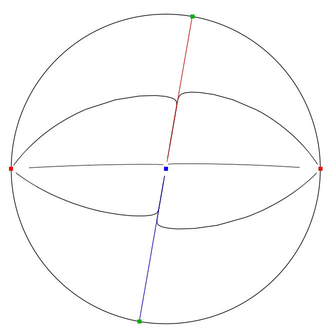

Figure 2. The phase portrait of system (5) in the Poincaré disc when $f = 0$. On the straight line $w = 0$ this phase portrait of the Szekeres system restricted to $F = 0$ and $H = h$ is not defined.

**Theorem 5.** *The global phase portraits of system* (5) *with* $f > 0$ *are topologically equivalent to the one described in Figure* 1*, and the global phase portraits of system* (5) *with* $f = 0$ *are topologically equivalent to the one described in Figure* 2.

*Proof.* There are four finite singular points of system (5) when $f > 0$ which are

$$p_1 = (0, \sqrt{f}),\ p_2 = \left(-\sqrt{f/3}\,h, \sqrt{f/3}\right),\ p_3 = \left(\sqrt{f/3}\,h, -\sqrt{f/3}\right),\ p_4 = (0, -\sqrt{f}).$$

Computing the eigenvalues of the Jacobian matrix evaluated at these points we get that $p_1$ is a stable node (the eigenvalues are $-8f^{3/2}, -f^{3/2}$), $p_2$ is an unstable node (the eigenvalues are $4f^{3/2}/\sqrt{3}, f^{3/2}/\sqrt{3}$), $p_3$ is a stable node (the eigenvalues are $-4f^{3/2}/\sqrt{3}, -f^{3/2}/\sqrt{3}$) and $p_4$ is an unstable node (the eigenvalues are $8f^{3/2}, f^{3/2}$).

On the other hand if $f = 0$ system (5) can be written as

$$\begin{aligned} \dot{z} &= w^3(-18z + 3hw), \\ \dot{w} &= \frac{3}{4}w^4, \end{aligned} \tag{6}$$

and introducing the reparameterization of time $ds = w^3\, dr$ we rewrite system (6) as

$$\begin{aligned} \dot{z} &= -18z + 3hw, \\ \dot{w} &= -\frac{3}{4}w, \end{aligned} \tag{7}$$

where now the dot means derivative with respect the new independent variable $r$. Note that the unique finite singular point of system (7) is the origin is a stable node because the eigenvalues of the Jacobian matrix at the origin are $-18$ and $-3/4$.

Now we study the infinite singular points when $f > 0$. We can work in both cases with system (5).

On the local chart $U_1$ system (5) writes

$$\begin{aligned} \dot{u} &= \frac{1}{4}(69u^4 - 36fu^2v^2 - f^2v^4 - 12hu^5 + 24fhu^3v^2 - 12f^2huv^4), \\ \dot{v} &= -v(-18u^3 + 10fuv^2 + 3hu^4 - 6fhu^2v^2 + 3f^2hv^4). \end{aligned} \tag{8}$$

The singular points at infinity in the local chart $U_1$ are $(u, v) = (0, 0)$ and $(u, v) = (23/(4h), 0)$. So, if $h \neq 0$ we have two singular points on $U_1$, and if $h = 0$ we have only the origin as a singular point on $U_1$.

If $h \neq 0$, computing the eigenvalues of the Jacobian matrix at $(23/(4h), 0)$ we get that they are $-839523/(256h^3)$ and $36501/(256h^3)$. So this point is a saddle.

On the other hand computing the Jacobian matrix at the origin $(0, 0)$ we get that it is identically zero. We need to do blow ups. We introduce the new variable $w = v/u$. In the new variables $(u, w)$ we can rewrite system (10) as

$$\begin{aligned} \dot{u} &= -\frac{u^4}{4}(-69 + 12hu + 36fw^2 - 24fhuw^2 + f^2w^4 + 12f^2huw^4), \\ \dot{w} &= \frac{u^3w}{4}(fw^2 - 3)(fw^2 - 1). \end{aligned} \tag{9}$$

Now doing a rescaling of the independent variable we eliminate the term $u^3$ in system (9) and we obtain

$$\begin{aligned} \dot{u} &= -\frac{u}{4}(-69 + 12hu + 36fw^2 - 24fhuw^2 + f^2w^4 + 12f^2huw^4), \\ \dot{w} &= \frac{w}{4}(fw^2 - 3)(fw^2 - 1). \end{aligned}$$

The solutions on $u = 0$ of $\dot{w} = 0$ are precisely $w = 0$, $w = -1/\sqrt{f}$, $w = 1/\sqrt{f}$, $w = -\sqrt{3/f}$ and $w = \sqrt{3/f}$. Computing the eigenvalues of the Jacobian matrix at these points we se that $(0, 0)$ is an unstable node (the eigenvalues are $69/4$, $3/4$),

$(0, \pm 1/\sqrt{f})$ are saddles (the eigenvalues are 8, $-1$ for both of them), $(0, \pm\sqrt{3/f})$ are saddles (the eigenvalues are $-12$, 3 for both of them).

Now going back through the changes of variables to system (10) we get that the origin of $U_1$ is formed by six hyperbolic sectors and two parabolic sectors. Three hyperbolic sectors are separated from the other three by the line at infinity and adjacent to the line at infinity in both sides there is a parabolic sector.

On the local chart $U_2$ system (5) becomes

$$\dot{u} = \frac{1}{4}(12h - 69u - 24fhv^2 + 36fuv^2 + 12f^2hv^4 + f^2uv^4),$$
$$\dot{v} = \frac{1}{4}v(fv^2 - 3)(fv^2 - 1).$$

The origin of $U_2$ is a singular point if and only if $h = 0$. In this last case computing the eigenvalues of the linear part of the differential system at the origin we get that they are $-69/4$ and $3/4$. So the origin of $U_2$ is a saddle.

Gluing all these information together we get that the global phase portrait of system (5) when $f > 0$ in the Poincaré disc $\mathbb{D}$ is topologically equivalent to the one of Figure 1.

Now we study the infinite singular points when $f = 0$. On the local chart $U_1$ system (5) writes

$$\dot{u} = \frac{3}{4}u^4(23 - 24hu), \qquad \dot{v} = 3u^3v(6 - hu). \tag{10}$$

Doing a rescaling of the independent variable we eliminate the term $u^3$ in (10) and we obtain

$$\dot{u} = \frac{3}{4}u(23 - 24hu), \qquad \dot{v} = 3v(6 - hu).$$

The singular points at infinity in the local chart $U_1$ are ((0, 0) and ((23/(4h), 0). So if $h \neq 0$ we have two infinite singular points on $U_1$, and if $h = 0$ we have only the origin as an infinite singular point on $U_1$.

If $h \neq 0$, computing the eigenvalues of the Jacobian matrix at $(23/(4h), 0)$ we get that they are $-69/4$ and $3/4$. So this point is a saddle. On the other hand, computing the eigenvalues of the Jacobian matrix at $(0, 0)$ we get that they are $69/4$ and 18. So this point is an unstable node.

On the local chart $U_2$ system (5) becomes

$$\dot{u} = \frac{3}{4}(4h - 23u), \qquad \dot{v} = \frac{3}{4}v.$$

The origin of $U_2$ is a singular point if and only if $h = 0$. In this last case computing the eigenvalues of the linear part of the differential system at the origin we get that they are $-69/4$ and $3/4$. So the origin of $U_2$ is a saddle.

Gluing all these information together we get that the global phase portrait of system (5) when $f = 0$ in the Poincaré disc $\mathbb{D}$ is topologically equivalent to the one of Figure 2. $\square$

Let $q \in \mathbb{D}$ and denote by $\phi_t(q)$ the solution of the extended flow in $\mathbb{D}$ of system (5) that at time $t = 0$ pass through the point $q$. We recall that a point $p \in \mathbb{D}$ is an

*$\omega$-limit* (resp. *$\alpha$-limit*) of a point $q \in \mathbb{D}$ if there are points $\phi_{t_1}(q), \phi_{t_2}(q), \ldots$ in the orbit of $q$ such that $t_k \to \infty$ (resp. $t_k \to -\infty$) and $\phi_{t_k}(q) \to p$ as $k \to \infty$.

*Proof of Theorems* 2 *and* 3. The proof of Theorem 2 comes interpreting the results provided in Theorem 5 on the gravitational Szekeres system (1) when $f > 0$, and the proof of Theorem 3 comes interpreting the results provided in Theorem 5 on the gravitational Szekeres system (1) when $f = 0$.

More precisely, we prove statement (a) of Theorem 2, the other statements as well as the proof of Theorem 3 are proved in a similar way. Consider the orbits over the hyperplane $w = \sqrt{f}$ for fixed values of $f > 0$ and $h \in \mathbb{R}$. From Figure 1 for the two fixed values of $f$ and $h$ there is a unique orbit $\gamma$ with $\alpha$-limit the infinite singular point $(23/(4h), 0) \in U_1$ if $h \neq 0$, or $(0,0) \in U_2$ if $h = 0$, and $\omega$-limit the finite stable node $(0, \sqrt{f})$. This orbit $\gamma$ for the two fixed values of $f$ and $h$ separates two kind of orbits. The ones with $\alpha$-limit the infinite singular point $(0,0) \in V_1$ and $\omega$-limit the finite stable node $(0, \sqrt{f})$, and the others with $\alpha$-limit the infinite singular point $(0,0) \in U_1$ and $\omega$-limit the finite stable singular point. This completes the proof of statement (a) of Theorem 2. $\square$

## 4. Conclusion

The gravitational Szekeres differential system (1) is completely integrable with two rational first integrals and an additional analytical first integral. One of these two rational first integrals is $F = (-18E - 3\rho + (\theta + 3\sigma)^2)^3/(6E + \rho)^2$.

The dynamics of the Szekeres system when the first integral $F$ takes negative values was described in [6], showing that all the orbits come from the infinity of $\mathbb{R}^4$ in the variables $(\rho, \theta, \sigma, E)$ and go to infinity. In other words all orbits of the gravitational Szekeres differential system born at infinity and end at infinity when the first integral $F < 0$.

In this paper we have described the dynamics when the first integral $F$ takes positive and zero values. Thus for $F > 0$ there are essentially three different types of orbits, ones that born at infinity and tend to a finite point of the phase space, others that born at a finite point of the phase space and end at infinity, and finally orbits that born and end in two distinct finite points of the phase space. For more precise information about these kind of orbits see Theorem 2.

If $F = 0$ then there are two types of orbits, the ones that born at infinity and tend to a finite point of the phase space, and the others that born at a finite point of the phase space and end at infinity. For more details on these orbits see Teorem 3.

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