POLYNOMIAL DIFFERENTIAL SYSTEMS WITH EVEN DEGREE HAVE NO GLOBAL CENTERS

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ABSTRACT. Let $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ be a differential system with P and Q real polynomials, and let $d = \max\{\deg P, \deg Q\}$. A singular point p of this differential system is a global center if $\mathbb{R}^2 \setminus \{p\}$ is filled with periodic orbits. We prove that if d is even then the polynomial differential systems have no global centers.

1. Introduction and statement of the main results

A singular point q of a vector field defined in $\mathbb{R}^2$ is a *center* if it has a neighbourhood filled of periodic orbits with the unique exception of q. The *period annulus* of the center q is the maximal neighbourhood U of q such that all the orbits contained in U are periodic except of course q. A center is *global* if its period annulus is $\mathbb{R}^2 \setminus \{q\}$. The notion of center goes back to Poincaré, see [6].

It is well known that any quadratic polynomial system (i.e. $n = 2$) has no global centers. The proof of this result is very large. It is based in classifying all the centers of the quadratic systems and then see that they are not global centers, see [2, 3, 4, 7, 8].

Let $P, Q \in \mathbb{R}[x, y]$ and $d = \max\{\text{deg}P, \text{deg}Q\}$. We will show that the polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

with d even do not have global centers. This is the main aim of this paper. Our proof for all d even is shorter than the existing one for $d = 2$.

Theorem 1. *The polynomial differential system* (1) *with even degree has no global centers.*

The proof of Theorem 1 is given in section 3.

In the following section we state and prove some auxiliary results that will be used during the proof.

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2. Auxiliary results

In the proof of Theorem 1 we will use the Poincaré compactification of a planar polynomial vector field $\mathcal{X}(x,y) = (P(x,y), Q(x,y))$ of degree d. The *Poincaré compactification* of $\mathcal{X}$, denoted by $p(\mathcal{X})$, is an induced vector field on $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$. We call $\mathbb{S}^2$ the *Poincaré sphere*. For more details on the Poincaré compactification see [5, Chapter 5]. Here we just introduce what will be needed.

Denote by $T_p\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at the point p. Assume that $\mathcal{X}$ is defined in the plane $T_{(0,0,1)}\mathbb{S}^2 = \mathbb{R}^2$. Consider the central projection $f\colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the open northern hemisphere $\mathcal{H}^+$ and other in the open southern hemisphere $\mathcal{H}^-$. Denote by $\mathcal{X}^1$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the infinity of $\mathbb{R}^2$. In order to extend $\mathcal{X}^1$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X}$ is a planar polynomial vector field of degree n then $p(\mathcal{X})$ is the only analytic extension of $y_3^{d-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \setminus \mathbb{S}^1 = \mathcal{H}^+ \cup \mathcal{H}^-$ there are two symmetric copies of $p(\mathcal{X})$, one in $\mathcal{H}^+$ and other in $\mathcal{H}^-$, and knowing the behaviour of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behaviour of $\mathcal{X}$ at infinity. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$. The singular points of $\mathcal{X}$ are called the *finite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$, while the singular points of $p(\mathcal{X})$ contained in $\mathbb{S}^1$, i.e. at infinity, are called the *infinite singular points* of $\mathcal{X}$ or of $p(\mathcal{X})$. It is known that the infinity singular points appear in pairs diametrically opposed.

To study the vector field $p(\mathcal{X})$ we use six local charts on $\mathbb{S}^2$ given by $U_k = \{y \in \mathbb{S}^2 : y_k > 0\}$, $V_k = \{y \in \mathbb{S}^2 : y_k < 0\}$ for $k = 1,2,3$. The corresponding local maps $\phi_k : U_k \to \mathbb{R}^2$ and $\psi_k : V_k \to \mathbb{R}^2$ are defined as $\phi_k(y) = -\psi_k(y) = (y_m/y_k, y_n/y_k)$ for $m < n$ and $m, n \neq k$. We denote by $z = (u,v)$ the value of $\phi_k(y)$ or $\psi_k(y)$ for any k, such that (u,v) will play different roles depending on the local chart we are considering. The points of the infinity $\mathbb{S}^1$ in any chart have $v = 0$. The expression for $p(X)$ in local chart (U_1, ϕ_1) is given by

$$\dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad \dot{v} = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right). \tag{2}$$

The infinite singular points are the endpoints of the straight lines defined by the real linear factors of the homogeneous polynomial $yP_d(x,y) - xQ_d(x,y)$, being P_d and Q_d the homogeneous parts of the polynomials P and Q of degree d.

Let q be an infinite singular point and let h be a hyperbolic sector of q. We say that h is *degenerate* if its two separatrices are contained in the equator of $\mathbb{S}^2$ (i.e. in $\mathbb{S}^1$). It is well-known that an infinite singular point

p formed by two degenerated hyperbolic sectors must have its linear part identically zero (see for instance Chapters 2,3 and Theorems 2.5, 2.19 and 3.5 of [5].

For proving Theorem 1 we will use the following proposition which characterizes when a polynomial differential system has a global center.

Proposition 2. *A polynomial vector field $\mathcal{X}(x,y) = (P(x,y), Q(x,y))$ without a line of singular points at infinity, has a global center if and only if it has a unique finite singular point which is a center and all the infinite singular points in the Poincaré sphere, if they exist, must be formed by two degenerated hyperbolic sectors.*

3. Proof of Theorem 1

It is well known that any homogeneous polynomial of degree d factorizes as

$$\prod_{i=1}^{r_1}(a_i x + b_i y)^{l_i} \prod_{k=0}^{r_2}(\alpha_k x^2 + \beta_k xy + \gamma_k y^2)^{j_k},$$

where $l_i \geq 0$ for all $i = 1, \ldots, r_1$, $j_k \geq 0$ and $\beta_k^2 - 4\alpha_k\gamma_k < 0$ for $k = 0, \ldots, r_2$ and $\sum_{i=1}^{r_1} l_i + \sum_{k=0}^{r_2} 2j_k = d$.

Let d_1 be the degree of P and d_2 be the degree of Q. We assume that $d = \max\{d_1, d_2\}$. The infinite singular points in the Poincaré disc of system (1) correspond to the linear factors of the quantity

$$G_d(x,y) = yP_d(x,y) - xQ_d(x,y) = 0$$

(it is well understood that P_d or Q_d could be zero).

We will separate the proof of Theorem 1 in two propositions dealing respectively with the cases $G_d \not\equiv 0$ and $G_d \equiv 0$. We start with the case $G_d \not\equiv 0$.

Proposition 3. *Any polynomial differential system* (1) *of degree d even and with $G_d \not\equiv 0$ do not have global centers.*

Proof. Taking into account that $G_d \not\equiv 0$, doing a rotation of the coordinate with respect to the origin we can assume that all the infinite singular points are in the local charts $U_1 \cup V_1$. We introduce the notation

$$G_{d-k}(x,y) = yP_{d-k}(x,y) - xQ_{d-k}(x,y) = 0, \quad k = 0, \ldots, d.$$

In the local chart U_1 system (1), using system (2), can be written as

$$\begin{aligned} \dot{u} &= -G_d(1,u) + vG_{d-1}(1,u) + \ldots + v^{d-1}G_0(1,0), \\ \dot{v} &= -vP_d(1,u) - v^2P_{d-1}(1,u) - \ldots - v^dP_0(1,u). \end{aligned} \tag{3}$$

The Jacobian matrix of any singular point $(\bar{u}, 0)$ of the local chart U_1 is of the form

$$\begin{pmatrix} \dfrac{\partial}{\partial u} G_d(1, \bar{u}) & G_{d-1}(1, \bar{u}) \\ 0 & -P_d(1, \bar{u}) \end{pmatrix}.$$

So the singular point $(\bar{u}, 0)$ if it exists (that is if $G_d(1, \bar{u}) = 0$) must be formed by two degenerate hyperbolic sectors, and as pointed out above it must be linearly zero. Hence $\dfrac{\partial}{\partial u} G_d(1, \bar{u}) = 0$. This implies that $G_d(1, \bar{u}) = 0$ and $\dfrac{\partial}{\partial u} G_d(1, \bar{u}) = 0$ and so the singular point $(\bar{u}, 0)$ must have multiplicity two as a zero of $G_d(1, u)$. This implies that if G_d has a real linear factor then it must have at least multiplicity two and so in general must be of the form (recall that G_d has degree $d + 1$)

$$G_d = \prod_{i=1}^{r_1} (a_i x + b_i y)^{l_i} \prod_{k=0}^{r_2} (\alpha_k x^2 + \beta_k xy + \gamma_k y^2)^{j_k}, \tag{4}$$

where $l_i \geq 2$ for all $i = 1, \ldots, r_1$, $j_k \geq 0$ and $\beta_k^2 - 4\alpha_k \gamma_k < 0$ for $k = 0, \ldots, r_2$ and $\sum_{i=1}^{r_1} l_i + \sum_{k=0}^{r_2} 2j_k = d + 1$.

Note that since $d+1$ is odd in (4), there exists at least $i \in \{1, \ldots, r_1\}$ and without loss of generality we can assume that it is $i = 1$, such that $l_1 \geq 3$ is odd. Then

$$G_d(x, y) = (a_1 x + b_1 y)^{l_1} \prod_{i=2}^{r_1} (a_i x + b_i y)^{l_i} \prod_{k=0}^{r_2} (\alpha_k x^2 + \beta_k xy + \gamma_k y^2)^{j_k}.$$

We can assume without loss of generality that $b_1 \neq 0$, otherwise we do a rotation with respect to the origin. Note that

$$G_d(1, u) = (a_1 + b_1 u)^{l_1} \prod_{i=2}^{r_1} (a_i + b_i u)^{l_i} \prod_{k=0}^{r_2} (\alpha_k + \beta_k u + \gamma_k u^2)^{j_k}.$$

Setting the new variable $a_1 + b_1 u = U$ (that is $u = (U - a_1)/b_1$ we have

$$G_d(1, U) = G_d\Big(1, \frac{U - a_1}{b_1}\Big) =: U^{l_1} \Gamma + \text{h.o.t.}, \tag{5}$$

where

$$\Gamma = \prod_{i=2}^{r_1} \Big(\frac{a_i b_1 - b_i a_1}{b_1}\Big)^{l_i} \prod_{k=0}^{r_2} \Big(\frac{\alpha_k b_1^2 - \beta_k a_1 b_1 + \gamma_k a_1^2}{b_1^2}\Big)^{j_k} \neq 0$$

(because $U = 0$ has exactly multiplicity l_1) and h.o.t means the higher order terms in the variable U. Taking the new variables (U, v), it follows from (3) and (5) that the system in the local chart U_1 restricted to $V = 0$ can be written as

$$\dot{U}|_{v=0} = (d + 1) U^{l_1} \Gamma + \text{h.o.t.}, \quad \dot{v}|_{v=0} = 0.$$

Note that the U-axis is invariant. In the positive semi-axis $\{U > 0, V = 0\}$ and in a neighborhood of $(U, V) = (0, 0)$ the orbit travels in the opposite sense to the orbit in the negative semi-axis $\{U < 0, V = 0\}$, and so the local phase portrait around $(U, V) = (0, 0)$ cannot be formed by two degenerated hyperbolic sectors. Hence, any Hamiltonian system (1) with n even and with H_{n+1} of the form (4) cannot have global centers. This concludes the proof of Proposition 3. □

Proposition 4. *A polynomial differential system* (1) *of degree d even and with $G_d \equiv 0$ has no global centers.*

Proof. Taking into account that the line at infinity is formed by singular points we must have that

$$G_d(x, y) \equiv 0 \quad \text{that is} \quad yP_d(x, y) \equiv xQ_d(x, y),$$

which implies that there exists a polynomial $R_d(x, y)$ of degree $d-1$ odd so that

$$P_d(x, y) = xR_d(x, y) \quad \text{and} \quad Q_d(x, y) = yR_d(x, y) \tag{6}$$

Note if system (1) has a global center then it has a unique finite singular point which is the origin and since the period annulus of that finite singular point is $\mathbb{R}^2 \setminus 0$, then the boundary of the period annulus U of the center of $p(\mathcal{X})$ located at $(0, 0, 1)$ is the equator of $\mathbb{S}^2$ or $\mathcal{H}^+$. Since there are no finite singular points in $\mathcal{H}^+$, except the center at $(0, 0, 1)$, and the infinite is formed by singular points, it follows that the boundary of the period annulus U is either a finite periodic orbit γ, or it is $\mathbb{S}^1$. If it is $\mathbb{S}^1$ then since it is formed by fixed points, then each singular point cannot be the ω-limit or de α-limit of any orbit. Now we show that it cannot be a finite periodic orbit γ. It it would be, we consider the Poincaré map π defined in a transversal section Π through γ. Since the vector field $p(\mathcal{X})$ is analytic, it follows that π is also analytic. Hence as π is the identity map in $\Pi \cap U$ it must be the identity map in $\Pi \cap (\mathcal{H}^+ \setminus U)$. But then the orbits in $\Pi \cap (\mathcal{H}^+ \setminus U)$ near U are also periodic, and γ is not the boundary of U, a contradiction.

In the local chart U_1 system (1), using system (2), can be written as

$$\begin{aligned} \dot{u} &= vG_{d-1}(1, u) + \ldots + v^{d-1}G_0(1, u), \\ \dot{v} &= -vR_d(1, u) - v^2P_{d-1}(1, u) - \ldots - v^dP_0(1, u). \end{aligned} \tag{7}$$

The line at infinity $v = 0$ if filled by singular points. We introduce the parameterization of time $ds = vdt$. With this new time system (7) becomes

$$\begin{aligned} \dot{u} &= G_{d-1}(1, u) + vG_{d-2}(1, u) + \ldots + v^{d-2}G_0(1, u), \\ \dot{v} &= -R_d(1, u) - vP_{d-1}(1, u) - \ldots - v^{d-1}P_0(1, u), \end{aligned} \tag{8}$$

where now the dot means derivative in the new time s.

Since $R_d(1, u) \not\equiv 0$, there exists $\overline{u}$ so that $R_d(1, \overline{u}) \neq 0$, and so a point $(\overline{u}, 0)$ is a regular point for system (8). Since $\dot{v}|_{v=0} = -R_d(1, u)$, and $\dot{v}|_{v=0, u=\overline{u}} \neq 0$ such point which is a singular point of system (8) would be the ω-limit or the $\alpha - limit$ of some orbit of system (7) and by the explanation above, system (1) cannot have a global center.

□

Proof of Theorem 1. The proof of Theorem 1 is an immediate consequence of Propositions 3 and 4. □

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References

[1] M.J. Álvarez, A. Ferragut and X. Jarque, *A survey on the blow up technique*, Internat. J. Bifur. Chaos **21** (2011), 3103–3118.
[2] J.C. Artés and J. Llibre, *Quadratic Hamiltonian vector fields*, Differential Equations **107** (1994), 80–95.
[3] J.C. Artés and J. Llibre, *Corrigendum: Quadratic Hamiltonian vector fields*, J. Differential Equations **129** (1996), 559–560.
[4] T. Date, *Classification and analysis of two-dimensional real homogeneous quadratic differential equation systems*, J. Differential equations **32** (1979), 311–334.
[5] F. Dumortier, J. Llibre and J. C. Artés, *Qualitative Theory of Planar Differential Systems*, Springer-Verlag, Berlin, 2006.
[6] H. Poincaré, *Mémoire sur les courbes définies par les équations différentielles*, Oeuvreus de Henri Poincaré, vol. 1, Gauthier–Villars, Paris, 1951, pp. 95–114.
[7] D. Schlomiuk, *Algebraic particular integrals, integrability and the problem of the center*, Trans. Amer. Math. Soc. **338** (1993), 799–841.
[8] N.I. Vulpe, *Affine invariant conditions for the topological distinction of quadratic systems with a center*, (Russian) Differentsialńye Uravneniya **19** (1983), 371–379; translation in Differential equations **19** (1983), 273–280.

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