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The zero-Hopf bifurcations of a four-dimensional hyperchaotic system

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ABSTRACT

We consider the four-dimensional hyperchaotic system $\dot{x} = a(y - x)$, $\dot{y} = bx + u - y - xz$, $\dot{z} = xy - cz$, and $\dot{u} = -du - jx + exz$, where a, b, c, d, j, and e are real parameters. This system extends the famous Lorenz system to four dimensions and was introduced in Zhou *et al.*, Int. J. Bifurcation Chaos Appl. Sci. Eng. **27**, 1750021 (2017). We characterize the values of the parameters for which their equilibrium points are zero-Hopf points. Using the averaging theory, we obtain sufficient conditions for the existence of periodic orbits bifurcating from these zero-Hopf equilibria and give some examples to illustrate the conclusions. Moreover, the stability conditions of these periodic orbits are given using the Routh–Hurwitz criterion.

I. INTRODUCTION

The chaos phenomenon is a complex dynamic behavior in a nonlinear dynamical system, which appears widely in nature. In 1963, meteorologist Lorenz[1] was the first to introduce the mathematical and physical chaotic model in $\mathbb{R}^3$, which is known as the Lorenz system. The Lorenz system planted the seed in chaos science. This system plays an important role in other areas such as in the modeling of lasers[2] and dynamos.[3] As one of the simplest models presenting chaos, the Lorenz system exhibits a rich range of dynamical properties, and it has been researched from different points of view, such as positive invariant,[4] integrability,[5–7] global dynamics,[8–10] and bifurcation.[11,12] After the Lorenz system, mathematicians and physicists from a physical or purely abstract mathematical point of view proposed various polynomial differential systems in $\mathbb{R}^3$, whose trajectories exhibit chaotic dynamics of the Lorenz system type. For example, one can refer to the Rikitake system,[13] Sprott A system,[14] Shimizu–Morioka system.[15]

Nowadays, three-dimensional nonlinear systems cannot provide adequate description of many phenomena in neural networks, social sciences, and engineering. To better describe the real world, we often necessitate to introduce high-dimensional (at least four dimensions) nonlinear systems. Recently, the hyperchaotic system has become a focus of research (see Refs. 16–23 and the references therein). The concept of hyperchaos was given by Rössler in Ref. 24. The precise definition of *hyperchaotic system* is as follows: (i) at least a four-dimensional autonomous differential system, (ii) a dissipative structure, and (iii) at least two unstable directions, of which at least one direction is nonlinear.[18] The hyperchaotic systems are very useful in secure communication due to the fact that the dynamic information of such systems is difficult to characterize and predict (see Ref. 25).

In this work, we use the classical averaging theory to investigate the zero-Hopf bifurcation of a hyperchaotic system. A zero-Hopf *equilibrium* is an equilibrium point of a four-dimensional autonomous differential system, which has a double zero eigenvalue and a pair of purely imaginary eigenvalues. There are rich works on three-dimensional zero-Hopf bifurcation (see Refs. 26–30). The zero-Hopf bifurcation of the hyperchaotic Lorenz system (i.e., four-dimensional) can be found in Refs. 17, 18, and 31. Actually, there are few results on the n-dimensional zero-Hopf bifurcation with $n > 3$.