# PERIODIC STRUCTURE OF THE TRANSVERSAL MAPS ON SURFACES

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Abstract. In this article we study the set of periods of transversal maps on orientable and non-orientable compact surfaces without boundary. We provide sufficient conditions, in terms of the spectra of the induced maps on homology, in order that the map has infinitely many periods, in particular odd periods.

## 1. Introduction and statements of the main results

Let $f$ be a continuous self–map on $X$. If $x \in X$ and $f(x) = x$ we say that $x$ is a *fixed point* of the map $f$. If $f^n(x) = x$ and $f^k(x) \neq x$ for all $k = 1, \ldots, n-1$, then we say that $x$ is a *periodic point* of the map $f$ of *period* $n$. We denote by Per(f) the set of the periods of all periodic points of a map $f : X \to X$.

Let $X$ be a $n$–dimensional topological manifold and $f$ a continuous self–map on $X$. The map $f$ induces a homomorphism on the $k$–th rational homology group of $X$ for $0 \leq k \leq n$, i.e. $f_{*k} : H_k(X, \mathbb{Q}) \to H_k(X, \mathbb{Q})$. The $H_k(X, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$ and $f_{*k}$ is a linear map whose matrix has integer entries.

The *Lefschetz number* of the map $f$ is an integer defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}). \tag{1}$$

The *Lefschetz Fixed Point Theorem* states that if $L(f) \neq 0$ then $f$ has a fixed point (*cf.* [2] or [12]).

The *Lefschetz numbers of period* $m$ are defined by

$$\ell(f^m) := \sum_{r|m} \mu(r) L(f^{m/r}), \tag{2}$$

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where the sum is taken over all divisors $r$ of $m$ and $\mu$ is the *Möbius function* defined by

$$\mu(m) := \begin{cases} 1 & \text{if } m = 1; \\ 0 & \text{if there is a } k \text{ such that } k^2 \text{ divides } m; \\ (-1)^s & \text{if } m = p_1 \cdots p_s \text{ with } p_i \text{ distinct primes.} \end{cases}$$

By the Möbius inversion formula we have

$$L(f^m) = \sum_{r|m} \ell(f^r).$$

The Lefschetz numbers of period $m$ were introduced in [4], see also [13, 1, 11]; for more recent developments in the characterization of these numbers see [8].

A *transversal map* $f$ on a compact differentiable manifold $X$ is a $C^1$ map $f : X \to X$, such that $f(X) \subset \mathrm{Int(X)}$ and for all positive integer $m$ at each point $x$ fixed by $f^m$ we have that 1 is not an eigenvalue of $Df^m(x)$, i.e. $\det(Id - Df^m(x)) \neq 0$.

The transversal maps have been studied in different contexts, see for example [3, 5, 6, 14, 20, 21]. In [9], it was studied transversal self-maps on spaces with the homology given by $H_0(X, \mathbb{Q}) = \mathbb{Q}$, $H_1(X, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$ and trivial for $H_k(X, \mathbb{Q})$ with $k \neq 0, 1$. In [14] it was considered the case of maps on the $n$-dimensional sphere and spaces with the same homology. In [7] the authors dealt with maps on the projective spaces and the product of two spheres of different dimensions. In [18] it was considered the periodic structure for transversal maps on the product of any given number of spheres of different dimension, provided that the set of partial sums of the dimensions of the spheres is a sum-free set, i.e. the homology spaces are either one-dimensional or trivial. Recently in [23] it was described the periodic structure of transversal self-maps on the $n$-torus, the product of spheres of the same dimension and on rational exterior spaces of a given rank.

We would like to remark that the Morse-Smale diffeomorphisms are transversal maps, since all their periodic points are hyperbolic. However they have only a finite number of periodic points. For a description of the periods set of Morse-Smale diffeomorphisms see the survey [17] and within references.

The periodic points for transversal maps on surfaces were initially studied in [19]. In particular they proved Theorem 12. In the present paper we improved this theorem by giving weaker hypotheses and the proofs are different because we used Lefschetz numbers of period $m$ (for some $m$'s) and Theorem 7. In section 4 we give a detail description of the differences of our results and the results of [19]. In section 2 we deal with transversal maps on non-orientable surfaces and in section 3, we deal with the case of orientable surfaces. The main results are the following ones.

**Theorem 1.** *Let $S_g$ be a non-orientable closed surface of genus $g$ and $f : S_g \to S_g$ be a transversal map. Let $\lambda_1, \dots, \lambda_{g-1}$ be the eigenvalues of $f_{*1}$, listed in descending order according the values of theirmodulus. Assume that*

(a) *Either $\lambda_1$ has multiplicity $k$ with $k \geq 1$ and $|\lambda_1| > |\lambda_j|$, for $k+1 \leq j \leq g-1$, i.e. the map $f_{*1}$ has a dominant eigenvalue.*
(b) *Or $\lambda_1$ has multiplicity $k \geq 2$ and $|\lambda_1| = |\lambda_i| > 1$ for $1 \leq i \leq k+l$ with $0 < l < k$.*

*Then there exists a positive integer $N$ such that*

(1) *If $m$ is odd and $m > N$ , then $m \in \mathrm{Per(f)}$.*
(2) *If $m$ is even and $m > N$, then $m$ or $m/2$ is in $\mathrm{Per(f)}$.*

(c) *Assume $\lambda_1$ has multiplicity $k$ and $|\lambda_1| = |\lambda_i| > 1$ for $1 \leq i \leq k+l$ with $l \geq k$, and $\lambda_i/\lambda_1$ are roots of unity for $k+1 \leq i \leq k+l$. Then there are infinitely many even $m$'s such that $m \in \mathrm{Per(f)}$ or $m/2 \in \mathrm{Per(f)}$.*

**Theorem 2.** *Let $M_g$ be an orientable closed surface of genus $g$ and $f : M_g \to M_g$ be a transversal map of degree $D$. Let $\lambda_1, \cdots, \lambda_{2g}$ be the eigenvalues of $f_{*1}$ counted with their multiplicities and listed in descending order according to the values of their modulus. Suppose that*

(i) *either $|D| > \max\{1, |\lambda_1|\}$, where $D$ is the degree of $f$,*
(ii) *or $f_{*1}$ has a dominant eigenvalue greater than the maximum of $1$ and $|D|$,*
(iii) *or $f_{*1}$ has a dominant eigenvalue with multiplicity greater than 1, and its modulus is equal to $|D| > 1$,*
(iv) *or $\lambda_1$ has multiplicity $k \geq 3$ and $|\lambda_1| = |D| = |\lambda_i| > 1$ for $1 \leq i \leq k+l$ with $0 < l < k-1$, and $|\lambda_1| > |\lambda_j|$ for $k+l \leq i \leq 2g$.*

*Then here exists a positive integer $N$ such that*

(a) *if $m > N$ is odd, then $m \in \mathrm{Per(f)}$.*
(b) *if $m > N$ is even, then $m$ or $m/2$ is in $\mathrm{Per(f)}$.*

(v) *If* $\lambda_1$ *has multiplicity* $k \geq 2$ *and* $|\lambda_1| = |D| = |\lambda_i| > 1$ *for* $1 \leq i \leq k+l$ *with* $l \geq k-1$, $\lambda_i/\lambda_1$ *are roots of unity for* $k+1 \leq i \leq k+l$, *and* $|\lambda_1| > |\lambda_j|$ *for* $k+l \leq i \leq 2g$.

*Then there are infinitely many even* $m$*'s such that* $m \in \mathrm{Per(f)}$ *or* $m/2 \in \mathrm{Per(f)}$.

Theorems 1 and 2 are proved in sections 2 and 3, respectively. For obtaining these results we prove that $\ell(f^m) \neq 0$ for $m$ sufficiently large, the techniques used in order to get these results are based on the techniques developed in [23].

The results of Theorems 1 and 2 are related with previous results of Llibre and Swanson [19]. In section 4 we compare our results with the ones of [19].

## 2. Transversal maps on non-orientable surfaces

A closed surface is a compact, connected and without boundary surface. We recall that $H_k(X, \mathbb{Q})$ is a vector space over $\mathbb{Q}$, so it is always torsion free. The homology spaces for non–orientable closed surfaces are $H_0(X, \mathbb{Q}) = \mathbb{Q}$, $H_2(X, \mathbb{Q}) = \{0\}$, and

$$H_1(X, \mathbb{Q}) = \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{g-1},$$

for the surface $X = S_g$ formed by the connected sum of $g \geq 1$ real projective planes, called a *non-orientable surface of genus* $g$. The well known examples are $S_1$ the projective plane and $S_2$ the Klein bottle.

Let $\lambda_1, \ldots, \lambda_{g-1}$ be the eigenvalues of $f_{*1}$, counted with heir multiplicities, we list them in descending order according to the values of their modulus:

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{g-1}| \geq 0.$$

As in the statement of Theorem 1 we say that $f_{*1}$ has a dominant eigenvalue if $\lambda_1$ has multiplicity $k \geq 1$, and $|\lambda_1| > |\lambda_i|$, for $k+1 \leq i \leq g-1$.

**Proposition 3.** *Let* $f : S_g \to S_g$ *be a continuous self map.*

(a) *If* $f_{*1}$ *has a dominant eigenvalue of modulus greater than* 1, *then there exists a positive* $N$ *such that* $\ell(f^m) \neq 0$ *for all* $m \geq N$.
(b) *If* $\lambda_1$ *has multiplicity* $k \geq 2$ *and* $|\lambda_1| = |\lambda_i| > 1$ *for* $1 \leq i \leq k+l$ *with* $0 < l < k$. *Then there exists a positive* $N$ *such that* $\ell(f^m) \neq 0$ *for* $m \geq N$.

(c) *If $\lambda_1$ has multiplicity $k$ and $|\lambda_1| = |\lambda_i| > 1$ for $1 \le i \le k+l$ with $l \ge k$ and $\lambda_i/\lambda_1$ are roots of unity for $k+1 \le i \le k+l$. Then there exists a positive $N$ such that $\ell(f^m) \ne 0$ for infinitely many even $m$'s with $m \ge N$.*

Before proving Proposition 3 we need the following auxiliary lemmas.

**Lemma 4.** *If $f_{*1}$ has a dominant eigenvalue $\lambda_1$ of modulus greater than 1, then there exist a positive integer $N_1$ and positive constants $C_1$ and $K_1$, such that $C_1|\lambda_1|^m \le |L(f^m)|$ for $m \ge N_1$; and $|L(f^m)| \le K_1|\lambda_1|^m$ for all $m \ge 1$.*

*Proof.* According to the definition of the Lefschetz numbers (1) we have

$$L(f^m) = 1 - \operatorname{trace}(f_{*1}^m) = 1 - \sum_{j=1}^{g-1} \lambda_j^m = \lambda_1^m \left( \frac{1}{\lambda_1^m} - \sum_{j=1}^{g-1} \left( \frac{\lambda_j}{\lambda_1} \right)^m \right).$$

Since $|\lambda_1| \ge |\lambda_i|$ for $2 \le i \le g-1$, it follows that

$$\left| \frac{1}{\lambda_1^m} - \sum_{j=1}^{g-1} \left( \frac{\lambda_j}{\lambda_1} \right)^m \right| \le g,$$

for all $m \ge 1$. So $K_1 = g$.

If $\lambda_1$ has multiplicity $k$, i.e. $\lambda_1 = \lambda_j$, for $1 \le j \le k$ and $|\lambda_1| > |\lambda_i|$ for $k+1 \le i \le g-1$, then there exist $N > 0$ and a positive constant $C_1$ such that

$$\left| \frac{1}{\lambda_1^m} - \sum_{j=1}^{g-1} \left( \frac{\lambda_j}{\lambda_1} \right)^m \right| = \left| \frac{1}{\lambda_1^m} - k - \sum_{j=k+1}^{g-1} \left( \frac{\lambda_j}{\lambda_1} \right)^m \right| \ge C_1$$

for $m > N$. This completes the proof. □

Observe that if we omit the hypothesis of a dominant eigenvalue, then Lemma 4 does not hold, for example consider $\lambda_1 = -\lambda_2$ and $\lambda_3 = 1$, then

$$L(f^m) = 1 - (\lambda_1^m + (-\lambda_1)^m + \lambda_3^m) = -\lambda_1^m(1 + (-1)^m),$$

i.e. $L(f^m) = 0$ for $m$ odd, but $L(f^m)$ is bounded away from zero, for $m$ even, when $|\lambda_1| \ge 1$. We handle this situation in the lemmas 5 and 6 which are weaker versions of Lemma 4.

**Lemma 5.** *Let $\lambda_1, \dots, \lambda_{g-1}$ be the eigenvalues of $f_{*1}$ so that $\lambda_1 = \lambda_j$, for $1 \le j \le k$ $|\lambda_i| = |\lambda_1| > 1$ and $\lambda_i \ne \lambda_1$ for $k+1 \le i \le k+l$, with $0 < l < k$, and $|\lambda_1| > |\lambda_i|$, with $k+l < i \le g-1$. Then there* *exists a positive integer $N_1$ and positive constants $C_1$ and $K_1$ such that $C_1|\lambda_1|^m \le |L(f^m)|$ for $m \ge N_1$, and $|L(f^m)| \le K_1|\lambda_1|^m$ for all $m$.*

*Proof.* The proof of $|L(f^m)| \le K_1|\lambda_1|^m$ for all $m$ is the same than in the proof of Lemma 4.

From (1) we have that

$$(3)\quad L(f^m) = 1 - \sum_{i=1}^{g-1} \lambda_i^m = \lambda_1^m \left( \frac{1}{\lambda_1^m} - k - \sum_{i=k+1}^{k+l} \frac{\lambda_i^m}{\lambda_1^m} - \sum_{i=k+l+1}^{g} \frac{\lambda_i^m}{\lambda_1^m} \right).$$

Since $|\lambda_i| = |\lambda_1|$ , for $k+1 \le i \le k+l$ we have

$$\left| \sum_{i=k+1}^{k+l} \frac{\lambda_i^m}{\lambda_1^m} \right| \le l.$$

It follows

$$\left| k + \sum_{i=k+1}^{k+l} \frac{\lambda_i^m}{\lambda_1^m} \right| \ge k - \left| \sum_{i=k+1}^{k+l} \frac{\lambda_i^m}{\lambda_1^m} \right| \ge k - l = C_2 > 0.$$

On the other hand there exists $N_1$ such that for $m \ge N_1$ we have that

$$\left| \frac{1}{\lambda_1^m} - \sum_{i=k+l+1}^{g} \frac{\lambda_i^m}{\lambda_1^m} \right| \le \frac{C_2}{2}.$$

Therefore $|L(f^m)\lambda_1^{-m}| \ge C_2/2 = C_1$ for $m \ge N_1$. □

We would like to remark that if $l \ge k$, the expression $k + \sum_{i=k+1}^{k+l} \frac{\lambda_i^m}{\lambda_1^m}$ could be arbitrary small or even equal 0 for infinitely many $m$'s, depending on the algebraic nature of the $\lambda_i$'s. For this reason we introduce the following lemma.

**Lemma 6.** *Let $\lambda_1, \dots, \lambda_{g-1}$ be the eigenvalues of $f_{*1}$ so that $\lambda_1 = \lambda_j$, for $1 \le j \le k$, $|\lambda_i| = |\lambda_1| > 1$ and $\lambda_i/\lambda_1$ are roots of unity, different from 1 for $k+1 \le i \le k+l$ with $l \ge k$, and $|\lambda_1| > |\lambda_i|$, for $k+l < i \le g-1$. Then there exists a positive integer $N_1$ and a positive constant $C_1$ such that $C_1|\lambda_1|^m \le |L(f^m)|$ for infinitely many even $m$'s with $m \ge N_1$.*

*Proof.* Let $a_m := \sum_{i=k+1}^{k+l} \frac{\lambda_i^m}{\lambda_1^m}$. By hypotheses the numbers $\lambda_i/\lambda_1$ are roots of unity. Therefore the sequence $\{a_m\}_m$ is periodic. So $(\lambda_i/\lambda_1)^m = 1$ for infinitely many even $m$'s. Since $k$ is a positive integer it yields the inequality

$$\left| k + \sum_{i=k+1}^{k+l} \frac{\lambda_i^m}{\lambda_1^m} \right| \ge C_2,$$

where $C_2$ is a positive constant. The rest of the proof follows using the same arguments of the proof of Lemma 5. □

In Lemma 6 the hypothesis of $\lambda_i/\lambda_1$ being roots of unity, is required since there are algebraic numbers of modulus equal 1, which are not root of unity, e.g. $(3+4i)/5$. Moreover there are algebraic integers of modulus equal 1, which are not roots of unity, e.g. some Galois conjugates of Salem numbers (*cf.* [22]).

*Proof ot Proposition* 3. From (2) we have

$$\ell(f^m) = L(f^m) + \sum_{\substack{r|m;\\ r\neq 1}} \mu(r) L(f^{m/r}),$$

so

$$|\ell(f^m)| \geq |L(f^m)| - |\sum_{\substack{r|m;\\ r\neq 1}} \mu(r) L(f^{m/r})|. \tag{4}$$

Let $\lambda_1$ be the dominant eigenvalue of $f_{*1}$. Let $m_1 = \max\{m/r : r|m,\, r \neq 1\}$, then we claim that

$$|\sum_{\substack{r|m;\\ r\neq 1}} \mu(r) L(f^{m/r})| \leq K_2 |\lambda_1|^{m_1(1+\varepsilon)}, \tag{5}$$

for some constant $K_2 > 0$, for all $m \geq N_2$ being $N_2$ a positive integer and $\varepsilon > 0$ arbitrary small.

Now we prove the claim. If $m$ is a prime number then $m_1 = 1$, then $\sum_{\substack{r|m;\\ r\neq 1}} \mu(r) L(f^{m/r}) = L(f)$. From the definition of the Lefschetz number $|L(f)| \leq C|\lambda_1|$ with $C = g$. Therefore $|L(f)| \leq C|\lambda_1|^{1+\varepsilon}$ for all $\varepsilon > 0$.

If $m$ is not prime, then $m = p_1^{a_1} \dots p_s^{a_s}$ for prime numbers $p_s > \dots > p_1 > 1$ and $a_i$ positive integers; so $m_1 = p_1^{a_1-1} p_2^{a_2} \cdots p_s^{a_s}$. It follows that

$$\sum_{\substack{r|m;\\ r\neq 1}} \mu(r) L(f^{m/r}) = \mu(p_1) L(f^{m_1}) + \sum_{\substack{r|m;\\ r\neq 1;\, r\neq p_1}} \mu(r) L(f^{m/r}).$$

Using Lemma 4 we have

$$\begin{aligned} |\sum_{\substack{r|m;\\ r\neq 1}} \mu(r)L(f^{m/r})| &\leq |L(f^{m_1})| + |\sum_{\substack{r|m;\\ r\neq 1,}} \mu(r)L(f^{m/r})| \\ &\leq C|\lambda_1|^{m_1} + \sum_{\substack{r|m;\\ r\neq 1,}} |\mu(r)||L(f^{m/r})| \\ &\leq C|\lambda_1|^{m_1} + C|\lambda_1^{m_1}|\sum_{\substack{r|m;\\ r\neq 1,}} |\mu(r)| \\ &\leq C|\lambda_1^{m_1}|(1+\sigma(m)). \end{aligned}$$

where $\sigma(m)$ is the number of divisors of $m$. It is known that for large $m$, $\sigma(m) \leq C'm^{\varepsilon}$, where $C'$ is a positive constant and $\varepsilon$ is arbitrary small, for more details see [10, Theorem 315, pp. 260]. Therefore, since $|\lambda_1| > 1$ there exist a positive constant $K_2$ and a positive integer $N_2$ such that $C|\lambda_1|^{m_1}(1+\sigma(m)) \leq K_2|\lambda_1|^{m_1(1+\varepsilon)}$, for all $m \geq N_2$. This proves the inequality (5).

By Lemma 4 there exists a constant $C_1 > 0$ and positive integer $N$ such that $|L(f^m)| \geq C_1|\lambda_1|^m$ for $m \geq N$. Hence from (4) and (5), it follows:

$$|\ell(f^m)| \geq C_1|\lambda_1|^m - K_2|\lambda_1|^{m_1(1+\varepsilon)} \geq |\lambda_1|^m(C_1 - K_2|\lambda_1|^{m_1(1+\varepsilon)-m}),$$

As we mentioned before if $m$ is prime then $m_1 = 1$, this implies that $\ell(f^m) \neq 0$ for large prime. If $m$ is a composite number then $m_1 \geq \sqrt{m}$, so $m_1(1+\varepsilon) - m \leq 0$ for $m$ sufficiently large. Hence $|\lambda_1|^{m_1(1+\varepsilon)-m}$ is arbitrary small for large $m$. Therefore $|\ell(f^m)| > 0$ for large $m$. This completes the proof of statement (a).

The proof of statements (b) and (c) follow similar arguments as the proof of statement (a) using Lemma 5 and 6 (respectively) instead of Lemma 4. □

The following theorem is one of the main results that relates the Lefschetz numbers of period $m$ with the set of periods of a transversal map.

**Theorem 7** ([3, 13, 9]). *Let $X$ be a compact manifold and $f : X \to X$ be a transversal map. Suppose $\ell(f^m) \neq 0$, for some $m$. Then*

(a) *If $m$ is odd, then $m \in$* Per(f).
(b) *If $m$ is even, then $m$ or $m/2$ is in* Per(f).

*Proof of Theorem* 1. It follows from Proposition 3 and Theorem 7. □

The *spectral radius of a linear transformation* $T$ is

$$\mathrm{sp}(T) := \max\{|\lambda| \,:\, \lambda \text{ eigenvalue of } T\}.$$

As a complement of Proposition 3 we present the following result.

**Proposition 8.** *Let* $f : S_g \to S_g$ *be a continuous self map such that* $sp(f_{*1}) \leq 1$*, then* $\ell(f^m) \neq 0$ *only for finitely many* $m$*.*

The proof of Proposition 8 is based on the fact that the sequence $\{L(f^m)\}$ is bounded (since $\mathrm{sp}(f_*1) \leq 1$) and the numbers $\ell(f^m)/m$ are integers (*cf.* [4]), for details see [1, Theorem 2.2].

The Morse-Smale diffeomorphisms are in the context of the previous proposition, because the eigenvalues of $f_{*1}$ are root of unity (see [24]) and they are transversal because all their periodic points are hyperbolic. The explicit computation of all posible numbers $\ell(f^m)$ for the Morse-Smale diffeomorphisms, can be found in [16] for non-orientable surfaces and in [15] for orientable surfaces. These results were obtained using the Lefschetz zeta function.

## 3. Transversal maps on orientable surfaces

The homology spaces for the closed orientable surfaces are $H_0(X, \mathbb{Q}) = \mathbb{Q}$, $H_2(X, \mathbb{Q}) = \mathbb{Q}$ and if $X = M_g$ for $g \geq 0$ an orientable surface of genus $g$, i.e. a sphere with $g$ handles, then

$$H_1(M_g, \mathbb{Q}) = \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{2g}.$$

If $f : M_g \to M_g$ is a continuous self map, then $f_{*2} = (D)$, where $D$ is the *degree* of $f$. Therefore, from (1) the Lefschetz numbers of $f$ are

$$L(f^m) = 1 - \mathrm{trace}(f_{*1}^m) + \mathrm{trace}(f_{*2}^m) = 1 - \sum_{i=1}^{2g} \lambda_i^m + D^m, \tag{6}$$

where $\lambda_1, \ldots, \lambda_{2g}$ are the eigenvalues of $f_{*1}$, we order them descendingly according to their modulus

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{2g}| \geq 0. \tag{7}$$

We need the following auxiliary result.

**Proposition 9.** *Let* $\rho := \max\{|\lambda_i|, |D|\}$ *and suppose that* $\rho > 1$ *and that* (7) *holds. Assume*

(i) *either* $\rho = |D| > |\lambda_1|$,
(ii) *or* $\rho = |\lambda_1| > |D|$,
(iii) *or* $\rho = |\lambda_1| = |D|$, *the multiplicity of* $\lambda_1$ *as eigenvalue of* $f_{*1}$ *is* $k \geq 1$ *and* $|\lambda_1| > |\lambda_i|$ *for* $k+1 \leq i \leq 2g$.
(iv) *or* $\rho = |\lambda_1| = |D|$, $\lambda_1$ *has multiplicity* $k \geq 3$ *and* $|\lambda_1| = |\lambda_i| > 1$ *for* $k+1 \leq i \leq k+l$ *with* $0 < l < k-1$.

*Then there exist a positive integer* $N_1$, *and positive constants* $C_1$ *and* $K_1$ *such that* $C_1\rho^m \leq |L(f^m)|$ *for* $m \geq N_1$; *and* $|L(f^m)| \leq K_1\rho^m$ *for all* $m \geq 1$.

(v) *If* $\rho = |\lambda_1| = |D|$, $\lambda_1$ *has multiplicity* $k \geq 1$ *and* $\lambda_i/\lambda_1$ *are roots of unity for* $k+1 \leq i \leq k+l$ *with* $l \geq k-1$.

*Then there exists a positive integer* $N_1$, *and positive constants* $C_1$ *and* $K_1$ *such that* $C_1\rho^m \leq |L(f^m)|$ *for infinitely many even* $m$*'s with* $m \geq N_1$; *and* $|L(f^m)| \leq K_1\rho^m$ *for all* $m$.

*Proof.* First notice that since $\rho > 1$ then

$$|L(f^m)| \leq 1 + \sum_{i=1}^{2g} |\lambda_i|^m + |D|^m \leq K_1\rho^m,$$

where $K_1 = 2g+2$ and $m \geq 1$.

If $\rho = |D| > |\lambda_1|$ then, by (9) we have:

$$|L(f^m)| = \left| \frac{1}{D^m} - \sum_{i=1}^{2g} \left(\frac{\lambda_i}{D}\right)^m + 1 \right| |D|^m.$$

Clearly there exist a positive integer $N_1$, and a positive integer $C_1$, such that $|L(f^m)|\rho^{-m} \geq C_1$ for $m \geq N_1$. The proposition follows under assumption (i).

If $\rho = |\lambda_1| > |D|$ then

$$\begin{aligned} |L(f^m)| &= \left| \frac{1}{\lambda_1^m} - \sum_{i=1}^{2g} \left(\frac{\lambda_i}{\lambda_1}\right)^m + \left(\frac{D}{\lambda_1}\right)^m \right| |\lambda_1|^m \\ &= \left| \frac{1}{\lambda_1^m} - k - \sum_{i=k+1}^{2g} \left(\frac{\lambda_i}{\lambda_1}\right)^m + \left(\frac{D}{\lambda_1}\right)^m \right| |\lambda_1|^m, \end{aligned} \tag{8}$$

where $k$ is the multiplicity of $\lambda_1$ as eigenvalue of $f_{*1}$. Clearly there exist a positive integer $N_1$ and a positive integer $C_1$ such that $|L(f^m)| \geq C_1\rho^m$ for $m \geq N_1$. The proposition follows under assumption (ii).

If $\rho = |\lambda_1| = |D|$ with $|\lambda_1| > |\lambda_i|$, for $k+1 \le i \le 2g$ and $k \ge 1$ the multiplicity of $\lambda_1$. Then from (8) there exist a positive integer $N_1$ and a positive integer $C_1$ such that $|L(f^m)| \ge C_1 \rho^m$, for $m \ge N_1$. The proposition follows under assumption (iii).

If $\lambda_1$ has multiplicity $k$ and $\rho = |\lambda_1| = |D| = |\lambda_i|$, for $1 \le i \le k+l$ and $|\lambda_1| > |\lambda_j|$, for $k+l \le j \le 2g$, with $l \ne k$ and $l < k-1$ it follows (9)

$$|L(f^m)| = \left| \frac{1}{\lambda_1^m} - k - \sum_{i=k+1}^{k+l} \left(\frac{\lambda_i}{\lambda_1}\right)^m - \sum_{i=k+l+1}^{2g} \left(\frac{\lambda_i}{\lambda_1}\right)^m + \left(\frac{D}{\lambda_1}\right)^m \right| |\lambda_1|^m.$$

By the same argument used in the proof of Lemma 5 there exists a positive integer $N_1$ and a positive integer $C_1$ such that $|L(f^m)|\rho^{-m} \ge C_1$, for $m \ge N_1$. Note that we require $l < k-1$, because the sum (9) have the term $(D/\lambda_1)^m$, which has modulus equal 1 and it is not in (3). The rest of the proof of the proposition under assumption (iv) follows from Lemma 5.

Similarly as in the proof of Lemma 6 there exists a positive integer $N_1$ and a positive integer $C_1$ such that $|L(f^m)|\rho^{-m} \ge C_1$ for $m \ge N_1$ and for infinitely many $m \ge N_1$. So the proof under assumption (v) follows from Lemma 6. □

The following result is the version of Proposition 3 for continuous maps on orientable surfaces.

**Proposition 10.** *Let $f : M_g \to M_g$ be a continuous self map satisfying* (7) *and*

(i) *either* $|D| > \max\{1, |\lambda_i| : 1 \le i \le 2g\}$,
(ii) *or* $|\lambda_1| > \max\{1, |D|\}$,
(iii) *or* $|\lambda_1| = |D| > 1$, *the multiplicity of* $\lambda_1$ *as eigenvalue of* $f_{*1}$ *is* $k > 1$ *and* $|\lambda_1| > |\lambda_i|$, *for* $k+1 \le i \le 2g$.
(iv) *or* $|\lambda_1| = |D| = |\lambda_i| > 1$ *for* $1 \le i \le k+l$ *and* $|\lambda_1| > |\lambda_j|$ *for* $k+l \le i \le 2g$, *with* $0 < l < k-1$ *where* $k$ *is the multiplicity of* $\lambda_1$ *as eigenvalue of* $f_{*1}$.

*Then there exists a positive integer $N$ such that $\ell(f^m) \ne 0$ for all $m \ge N$.*

(v) *If* $|\lambda_1| = |D| = |\lambda_i| > 1$ *for* $1 \le i \le k+l$ *and* $\lambda_i/\lambda_1$ *are roots of unity for* $k+l \le i \le 2g$, *with* $l \ge k-1$ *where* $k$ *is the multiplicity of* $\lambda_1$ *as eigenvalue of* $f_{*1}$.

*Then there exists a positive integer $N$ such that $\ell(f^m) \neq 0$, for infinitely many even $m$'s with $m \geq N$.*

*Proof.* Since the proof goes to the same lines as the proof of Proposition 3 using Proposition 9 instead of Lemma 4 we only provide the main new steps of it.

Let $\rho = \max\{|\lambda_1|, |D|\}$. Suppose that one of the conditions (i), (ii) or (iii) holds. As in the proof of Lemma 4, we have inequality (4) and for large $m$'s

$$|\sum_{\substack{r|m;\\ r\neq 1}} \mu(r) L(f^{m/r})| \leq K_3 \rho^{m_1(1+\varepsilon)},$$

where $m_1 = \max\{m/r \,:\, r|m,\, r \neq 1\}$, $K_3$ is a positive constant and $\varepsilon > 0$ is sufficiently small. By Proposition 9 under one of the assumptions (i), (ii) or (iii) there exists a constant $C > 0$ and a positive integer $N$ such that $|L(f^m)| \geq C\rho^m$ for $m \geq N$. Therefore

$$|\ell(f^m)| \geq C\rho^m - K_3\rho^{m_1(1+\varepsilon)} \geq \rho^m(C - K_3\rho^{m_1(1+\varepsilon)-m}).$$

By the same argument as in the proof of Lemma 4, it follows that $|\ell(f^m)| > 0$ for large $m$'s.

However if $\rho = |\lambda_1| = |D| = |\lambda_i| > 1$ for $1 \leq i \leq k+l$ and $|\lambda_1| > |\lambda_j|$ for $k+l \leq i \leq 2g$, with $0 < l < k-1$ where $k$ is the multiplicity of $\lambda_1$ as eigenvalue of $f_{*1}$. Then, by Proposition 9 under assumption (iv) there exist a constant $C > 0$ and a positive integer $N$ such that $|L(f^m)| \geq C\rho^m$ for $m \geq N$. Therefore in this case $|\ell(f^m)| > 0$, for $m \geq N$.

Similarly, under assumption (v), by Lemma 6 there exist a constant $C > 0$ and a positive integer $N$ such that $|L(f^m)| \geq C\rho^m$ for infinitely many $m$'s with $m \geq N$. Hence $|\ell(f^m)| > 0$ for infinitely many $m$'s $m \geq N$. □

*Proof of Theorem* 2. It follows from Propositions 9 and 10 and Theorem 7. □

Similarly to Proposition 8 in the context of orientable surfaces we have the following result.

**Proposition 11.** *Let $f : M_g \to M_g$ be a continuous self map. If $\max\{sp(f_{*1}), |D|\} \leq 1$, then $\ell(f^m) \neq 0$ only for finitely many $m$.*

## 4. Remarks

Here we will give a detail description of the differences of our results and the previous results of Llibre-Swanson [19]. Before we recall some definitions. We say that a subset $B$ of $A$ is *cofinite* if $A \setminus B$ is finite. A transversal map $f$ is *semi-positive* if at each $x$, fixed by $f^m$, we have $\det(Df^m(x)) \geq 0$.

The main theorem of [19] is the following one.

**Theorem 12.** *Let $f : X \to X$ be a transversal map defined on a surface $X$. Assume that $f$ verifies either $D > \max\{1, sp(f_{*1})\}$, or $sp(f_{*1}) > \max\{1, D\}$ and $\lim_{m\to\infty} |\mathrm{trace}(f_{*1}^m)|^{1/m}$ exists. Then all the following statements holds.*

(1) Per(f) *contains a cofinite subset of odd positive integers.*
(2) Per(f) *contains infinitely many powers of two. Furthermore, there exists a positive integer $m$ such that two consecutive powers of two larger than $m$ include (at least) one period of $f$.*
(3) *Moreover, if $f$ is semi-positive, then* Per(f) *contains a cofinite subset of the set of all powers of two.*

We would like to emphasize that our methods are different from the ones used in [19]. As the reader have noticed we use the Lefschetz numbers of period $m$. In particular we show that $\ell(f^m) \neq 0$ for large $m$'s, and later we use Theorem 7.

First we comment the case of transversal maps on orientable surfaces. The hypotheses (i) and (ii) of Theorem 2 imply that the $\mathrm{sp}(f_{*1}) > 1$ and the existence of $\lim_{m\to\infty} |\mathrm{trace}(f_{*1}^m)|^{1/m}$. However the assumptions (iii), (iv) and (v) of Theorem 2 are not covered in Theorem 12.

In the case of transversal maps on non-orientable surfaces. The hypotheses of statements (a) and (b) of Theorem 1 imply the $\mathrm{sp}(f_{*1}) > 1$ and the existence of $\lim_{m\to\infty} |\mathrm{trace}(f_{*1}^m)|^{1/m}$. So we can use in this situation Theorem 12. Again the result of statement (c) of Theorem 1 is not covered by Theorem 12.

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