Arch. Math., Vol. 39, 331–334 (1982)

0003-889X/82/3904-0331 $ 2.30/0

On the structure of the set of periodic points of a continuous map of the interval with finitely many periodic points

By

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1. Introduction. Let I denote a closed interval on the real line and let $C^0(I, I)$ denote the space of continuous maps from I into itself. For $f \in C^0(I, I)$ let $P(f)$ denote the set of positive integers k such that f has a periodic point of period k (see section 2 for definitions).

From Šarkovskii's theorem we know: (i) if $P(f)$ is finite then $P(f) = \{1, 2, 4, \ldots, 2^n\}$ for some integer $n \geqq 0$ (see [3], [4] or [5]), (ii) if P is a periodic orbit of f of period 2^m, then f has a periodic orbit of period 2^k, which is contained in [min P, max P] for each $k = 0, 1, \ldots, m - 1$ (see [5]). In this paper we study the relation between these orbits.

Our main result is the following.

Theorem A. *Let $f \in C^0(I, I)$ and suppose $P(f) = \{1, 2, 4, \ldots, 2^n\}$. Then for any periodic orbit of period 2^m, with $m \leqq n$, there exist $m + 1$ periodic orbits of periods $1, 2, 4, \ldots, 2^m$ such that the 2^k periodic points of period 2^k are separated by the $1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1$ fixed points of $f^{2^{k-1}}$, for any $k = 1, 2, \ldots, m$ (see Fig. 1 for $m = 3$).*

This theorem will be proved in section 3.

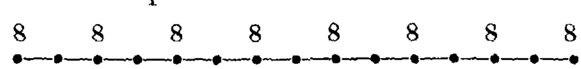

Figure 1.

Let $f \in C^0(I, I)$ and suppose $P(f) = \{1, 2, 4, \ldots, 2^n\}$. If f has a unique periodic orbit of period 2^k for any $k = 0, 1, \ldots, n$ (for instance the map given by Block in [1]), then Theorem A give us the complete structure of the set of periodic points of f.

2. Preliminary definitions and results. Let $f \in C^0(I, I)$. For any positive integer n, we define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. We let f^0 denote the identity map of I.

Let $p \in I$. We say p is a fixed point of f if $f(p) = p$. If p is a fixed point of f^n, for some $n \in N$ (the set of positive integers), we say p is a periodic point of f. In this case, the smallest element of $\{n \in N : f^n(p) = p\}$ is called the period of p.