RATIONAL FIRST INTEGRALS OF THE LIÉNARD EQUATIONS: THE SOLUTION TO THE POINCARÉ PROBLEM FOR THE LIÉNARD EQUATIONS

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Abstract. Poincaré in 1891 asked about the necessary and sufficient conditions in order to characterize when a polynomial differential system in the plane has a rational first integral. Here we solve this question for the class of Liénard differential equations $\ddot{x} + f(x)\dot{x} + x = 0$, being $f(x)$ a polynomial of arbitrary degree. As far as we know it is the first time that all rational first integrals of a relevant class of polynomial differential equations of arbitrary degree has been classified.

1. The Poincaré problem on the rational first integrals of the polynomial differential systems

A *rational function* $f(x,y)/g(x,y)$ has degree m if the polynomials $f(x,y)$ and $g(x,y)$ are coprime in the ring $\mathbb{R}[x,y]$, and the maximum of the degrees of $f(x,y)$ and $g(x,y)$ is m.

A *polynomial differential system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{x} = Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y, and t is the independent variable usually called the *time*. The *polynomial vector field* associated to the polynomial differential system (1) is

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

Let U be an open subset of $\mathbb{R}^2$. Here a *first integral* is a $\mathcal{C}^1$ non–locally constant function $H : U \to \mathbb{R}$ such that it is constant on the solutions $(x(t), y(t))$ of the polynomial differential system (1) contained in U, i.e. if $\mathcal{X}(H)|_U \equiv 0$.

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If the function H is rational then we say that H is a *rational first integral*.

The problem of providing necessary and sufficient conditions in order that a polynomial differential system in the plane has a rational first integral was stated by Poincaré in 1891 in [12]. This problem is of a global nature involving whole classes of polynomial differential systems and this is one of the reasons for being so hard.

If $\mathcal{X}$ is a polynomial vector field on $\mathbb{R}^2$ the *n-th extactic curve of* $\mathcal{X}$, $\mathcal{E}_n(\mathcal{X})$, is defined by the polynomial equation

$$\det\begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \cdots & \mathcal{X}(v_l) \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{X}^{l-1}(v_1) & \mathcal{X}^{l-1}(v_2) & \cdots & \mathcal{X}^{l-1}(v_l) \end{pmatrix} = 0,$$

where $v_1, v_2, \cdots, v_l$ is a basis of $\mathbb{R}_n[x,y]$, the $\mathbb{R}$–vector space formed by all polynomials in $\mathbb{R}[x,y]$ of degree at most n, and so $l = (n+1)(n+2)/2$, and $\mathcal{X}^j(v_i) = \mathcal{X}^{j-1}(\mathcal{X}(v_i))$. Observe that the definition of extactic curve is independent of the chosen basis of the $\mathbb{R}$–vector space of polynomials of degree at most n.

As far as we know the first solution of this problem was given in the next result.

Theorem 1. *Let $\mathcal{X}$ be a polynomial vector field. Then the polynomial $\mathcal{E}_n(\mathcal{X})$ is identically zero and the polynomial $\mathcal{E}_{n-1}(\mathcal{X})$ is not identically zero if, and only if, $\mathcal{X}$ admits a rational first integral of degree n.*

This result is Theorem 4.3 of the paper [2]. But in general Theorem 1 is difficult to apply because if the degree of the rational first integral is higher, then the computation of the determinant which appears in the definition of $\mathcal{E}_n(\mathcal{X})$ is not easy.

2. The solution to the Poincaré problem for the Liénard equations

One of the more studied classes of polynomial differential equations are the *Liénard differential equations*, or simply *Liénard equations*

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{2}$$

where $f(x)$ is a polynomial. The first in considering the differential equations of the form (2) was Liénard [7] during the development of radio and vacuum tube technology. Later on these equations were intensely studied as they can be used to model oscillating circuits, see for instance the classical books [1, 13, 14, 15].

Passing to the Liénard plane the second order differential equation (2) is equivalent to the first order polynomial differential system

$$\dot{x} = y - F(x), \qquad \dot{y} = -x, \tag{3}$$

where $F(x) = \displaystyle\int_0^x f(s)ds$.

Another way to write the second order differential equation (2) as a planar differential system of first order is

$$\dot{x} = y, \qquad \dot{y} = -f(x)y - x. \tag{4}$$

The objective of this paper is to solve the problem stated by Poincaré on the existence of rational first integrals for the class of polynomial Liénard differential systems (3), and consequently also for the equivalent classes of differential equations (2) and (4).

Consider the polynomial differential systems (3) in $\mathbb{R}^2$ where $F(x) = F_n(x)$ is polynomial in x of degree $n \geq 1$. These differential systems are called simply *Liénard systems.*

We denote by

$$\mathcal{X} = (y - F_n(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

the polynomial vector field associated to system (3).

Our main results are the following three theorems.

Theorem 2. *Liénard systems* (3) *of degree* 1 *has no rational first integral of degree* 1*.*

The *Totiente Euler function* $\phi(x)$ is such that for each $x \in \mathbb{N} = \{1, 2, 3, \ldots\}, \phi(x)$ is the quantity of numbers $k \in \{1, 2, \ldots, x\}$ such that $(k, x) = 1$, that is x and k are relatively prime.

$$\phi(x) = \#\{n \in \mathbb{N} : n \leq x \wedge (n, x) = 1\}.$$

The fundamental theorem of arithmetic states that if $x > 1$ there is a unique expression for $x = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $1 < p_1 < p_2 < \cdots < p_r$ are prime numbers and each integer $k_i \geqslant 1$. Then the function $\phi(x)$ has following expression

$$\phi(x) = x\prod_{p|x}\left(1 - \frac{1}{p}\right),$$

where the product is over the distinct prime numbers dividing x, for more details on the Totiente Euler function see Theorem 62 of [8].

Theorem 3. *For all $m \geq 2$ there are $2\phi(m)$ Liénard systems* (3) *of degree* 1 *with a rational first integral of degree m.*

Theorem 4. *There are not Liénard systems* (3) *of degree* > 1 *having rational first integrals.*

The proofs of Theorems 2, 3 and 4 are proved in the next section.

We note that Theorems 2, 3 and 4 characterize all the Liénard equations (2) which have rational first integrals. As far as we know it is the first time that all rational first integrals of a relevant class of polynomial differential equations of arbitrary degree has been classified.

We remark that the limit cycles of the Liénard equations (2) has been intensively studied, see for instance [3, 4, 5, 6, 10, 11]. But it remains many open questions about these limit cycles.

3. Proof of the results

Proof of Theorem 2. Consider system (3) with $n = 1$ and $\mathcal{X}$ its associated vector field. Then the 1-th extactic polynomial of $\mathcal{X}$, $\mathcal{E}_1(\mathcal{X})$, is

$$\det \begin{pmatrix} 1 & x & y \\ 0 & \mathcal{X}(x) & \mathcal{X}(y) \\ 0 & \mathcal{X}^2(x) & \mathcal{X}^2(y) \end{pmatrix} = -x^2 - (a_0 - y)(a_0 - y + a_1 x).$$

Since the polynomial $\mathcal{E}_1(\mathcal{X})$ is not identically zero, by Theorem 1, the Liénard systems (3) of degree 1 have no rational first integrals of degree 1. $\square$

Proof of Theorem 3. We consider the Liénard system of degree 1 given by

$$\dot{x} = y - a_0 - a_1 x, \quad \dot{y} = -x. \tag{5}$$

It is easy to check that system (5) is integrable with the first integral

$$\begin{aligned} H \; = \; & \left(a_1 \left(\sqrt{\frac{a_1^2 - 4}{a_1^2}} - 1 \right) (a_0 + a_1 x - y) + 2x \right)^{1 - \frac{a_1^2}{2}\left(\sqrt{1 - \frac{4}{a_1^2}} + 1\right)} \\ & \left(a_1 \left(\sqrt{\frac{a_1^2 - 4}{a_1^2}} + 1 \right) (a_0 + a_1 x - y) - 2x \right). \end{aligned}$$

So in order that from the expression of the function H we can obtain rational first integrals of degree m we must have

$$1-\frac{a_1^2}{2}\left(\sqrt{1-\frac{4}{a_1^2}}+1\right)=-\frac{m}{r}, \tag{6}$$

withe $r\in\{1,\dots,m-1\}$ and $(r,m)=1$. Solving equations (6) with respect to a_1 we get

$$a_1=\pm\frac{m+r}{\sqrt{mr}}.$$

So, for a given positive integer $m>1$ we have $\phi(m)$ good numbers m/r for which from the expression of H we can obtain $\phi(m)$ different Liénard systems of degree 1 with a rational first integral of degree m. This completes the proof of the theorem. ☐

From Theorem 3 we provide the explicit Liénard systems with a rational first integral H of degree $2,3$ and 4.

Example 5. *Consider $m=2$. From Theorem 3 there are two Liénard systems of degree 1 with H a rational first integral of degree 2, which are given by:*

For $r=1$,

$$\dot{x}=y-a_0+3x/\sqrt{2},\quad \dot{y}=-x,\quad \textit{with}\quad H=\frac{-\sqrt{2}a_0+2x+\sqrt{2}y}{\left(-\sqrt{2}a_0+x+\sqrt{2}y\right)^2},$$

$$\dot{x}=y-a_0-3x/\sqrt{2},\quad \dot{y}=-x,\quad \textit{with}\quad H=\frac{\sqrt{2}a_0+2x-\sqrt{2}y}{\left(\sqrt{2}a_0+x-\sqrt{2}y\right)^2}.$$

Example 6. *Consider $m=3$. From Theorem 3 there are four Liénard systems of degree 1 with H a rational first integral of degree 3, which are given by:*

For $r=1$,

$$\dot{x}=y-a_0+4x/\sqrt{3},\quad \dot{y}=-x,\quad \textit{with}\quad H=-\frac{-\sqrt{3}a_0+3x+\sqrt{3}y}{\left(-\sqrt{3}a_0+x+\sqrt{3}y\right)^3},$$

$$\dot{x}=y-a_0-4x/\sqrt{3},\quad \dot{y}=-x,\quad \textit{with}\quad H=-\frac{\sqrt{3}a_0+3x-\sqrt{3}y}{\left(\sqrt{3}a_0+x-\sqrt{3}y\right)^3}.$$

For $r=2$,

$$\dot{x}=y-a_0+5x/\sqrt{6},\quad \dot{y}=-x,\quad \textit{with}\quad H=\frac{\left(-2a_0+\sqrt{6}x+2y\right)^2}{\left(\sqrt{6}a_0-2x-\sqrt{6}y\right)^3},$$

$$\dot{x} = y - a_0 - 5x/\sqrt{6}, \quad \dot{y} = -x, \quad \textit{with} \quad H = \frac{\left(2a_0 + \sqrt{6}x - 2y\right)^2}{\left(-\sqrt{6}a_0 - 2x + \sqrt{6}y\right)^3}.$$

Example 7. *Consider $m = 4$. From Theorem 3 there are four Liénard systems of degree 1 with H a rational first integral of degree 4, which are given by:*

For $r = 1$,

$$\dot{x} = y - a_0 + 5x/2, \quad \dot{y} = -x, \quad \textit{with} \quad H = \frac{-a_0 + 2x + y}{(-2a_0 + x + 2y)^4},$$

$$\dot{x} = y - a_0 - 5x/2, \quad \dot{y} = -x, \quad \textit{with} \quad H = \frac{a_0 + 2x - y}{(2a_0 + x - 2y)^4}.$$

For $r = 3$,

$$\dot{x} = y - a_0 + 7x/\sqrt{12}, \quad \dot{y} = -x, \quad \textit{with} \quad H = \frac{\left(-\sqrt{3}a_0 + 2x + \sqrt{3}y\right)^3}{\left(2\sqrt{3}a_0 - 3x - 2\sqrt{3}y\right)^4},$$

$$\dot{x} = y - a_0 - 7x/\sqrt{12}, \quad \dot{y} = -x, \quad \textit{with} \quad H = \frac{\left(\sqrt{3}a_0 + 2x - \sqrt{3}y\right)^3}{\left(-2\sqrt{3}a_0 - 3x + 2\sqrt{3}y\right)^4}.$$

To prove Theorem 4 we use the following result. In 1996 Hayashi [9] studied the invariant algebraic curves for the Liénard system

$$\dot{x} = y, \qquad \dot{y} = -f(x)y - g(x), \tag{7}$$

where f and g are polynomials of degree M and N respectively and obtained the following result.

Theorem 8. *Under the conditions $f(x) \neq 0$, and $M + 1 \geq N$ the Liénard system (7) has an invariant algebraic curve if and only if there is an invariant curve $y = P(x)$ satisfying*

$$g(x) = -\left[f(x) + P'(x)\right] P(x),$$

where $P(x)$ or $P(x) + \int f(x)dx$ is a polynomial of degree at most one.

Proof of Theorem 4. Consider system (3) of degree $n > 1$, with $F(x) = \sum_{i=0}^{n} a_i x^i$ and $a_n \neq 0$.

System (3) is equivalent to system (7) if $f(x) = F'(x)$ and $g(x) = x$. Therefore by Theorem 8 system (3) has an invariant algebraic curve if and only if

(i) $P(x) = d_0 + d_1 x$, and

$$g(x) = -\left[f(x) + P'(x)\right] P(x) \quad \Leftrightarrow \quad x = -\left[f(x) + d_1\right](d_0 + d_1 x). \tag{8}$$

Then from last equation we obtain that $-na_n d_1 x^n = 0$, a contradiction if $d_1 \neq 0$. If $d_1 = 0$ then get $-na_n d_0 x^{n-1} = 0$, a contradiction $d_0 \neq 0$. If $d_1 = d_0 = 0$ then we obtain a contradiction in the last equation of (8).

(ii) $P(x) = d_0 + d_1 x - F(x)$, and

$$g(x) = -\left[f(x) + P'(x)\right] P(x) = \quad \Leftrightarrow \quad x = -d_1(d_0 + d_1 x - F(x)). \tag{9}$$

From the last equation we have thet $d_1 a_n x^n = 0$, a contradiction if $d_1 \neq 0$. If $d_1 = 0$ then again we have a contradiction in the last equation (9).

Therefore system (3) with $n > 1$ has no invariant algebraic curves, so systems (3) cannot have rational first integrals. □

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