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Polynomial Vector Fields on the Clifford Torus

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First, we characterize all the polynomial vector fields in $\mathbb{R}^4$ which have the Clifford torus as an invariant surface. Then we study the number of invariant meridians and parallels that such polynomial vector fields can have on the Clifford torus as a function of the degree of these vector fields.

Keywords: Invariant parallels; invariant meridians; polynomial vector field; Clifford torus.

1. Introduction and Statement of the Main Results

The Clifford torus

$$\mathbb{T} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = \frac{1}{2}, x_3^2 + x_4^2 = \frac{1}{2} \right\}$$

in geometric topology is the simplest and most symmetric Euclidean space embedding of the cartesian product of two circles. It lives in $\mathbb{R}^4$, as opposed to $\mathbb{R}^3$.

In MathSciNet on May 21, 2020, the keyword "Clifford torus" appears with 489 references. The more recent reference is [Li & Wang, 2017]. In [Dursun & Turgay, 2013] the authors studied the meridians of the surfaces of revolution and some information on the meridians of the Clifford torus can also be found there. In [de Barros *et al.*, 2004; Zhuang & Zhang, 2011] the authors studied the parallels of the surfaces of revolutions, containing some information on the parallels of the Clifford torus.

In this paper, we shall first study the polynomial vector fields of arbitrary degree in $\mathbb{R}^4$ having the Clifford torus invariant by their flow, and later, we shall compute the maximal number of parallels and meridians that a polynomial vector field of a given degree can exhibit on the Clifford torus.

The maximum number of invariant hyperplanes that a polynomial vector field in $\mathbb{R}^n$ can have as a function of its degree was given in [Llibre & Medrado, 2007]. The analogous result for the invariant straight lines of polynomial vector fields in $\mathbb{R}^2$ was provided before in [Artés *et al.*, 1998]. The study of the maximum number of meridians and parallels for a torus in $\mathbb{R}^3$ were studied in [Llibre & Medrado, 2011], and for an algebraic torus in [Llibre & Rebollo, 2013]. In surfaces of revolution in $\mathbb{R}^3$, the meridians and parallels invariant by polynomial vector fields have been studied in [Dias *et al.*, 2016].

As usual, we denote by $\mathbb{R}[x_1, x_2, x_3, x_4]$ the ring of the polynomials in the variables x_1, x_2, x_3 and x_4 with real coefficients. By definition, a *polynomial differential system in* $\mathbb{R}^4$ is a system of the