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On sizes and shapes of lattice figures

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In this paper we study the size of a kind of lattice figures in the Euclidean space $\mathbb{R}^n$, called the L-hyperpolyhedra. We emphasize the relationship between the size and the shape of such figures through the Euler characteristic. We obtain formulae for computing the volume of any L-hyperpolyhedron. Furthermore, we give plausible arguments for showing that these formulae also work for computing the volume of a more general type of figures (called the R-hyperpolyhedra). The article depicts a gradual evolution in the mathematical solution of the computation of the volume of an L-hyperpolyhedron. We follow a heuristic approach, which shows how some new problems and concepts appear as well as the role of intuition in the statement of conjectures. In this way, it claims to reveal some aspects of the didactical significance of mathematics as a process instead of as a closed set of results.

1. Introduction

Our main goal is the study of the volume (size) of lattice polyhedra and its relation with their topology (shape). We do not want to go directly to the proofs of the formulae which give the volume of the lattice polyhedra taking into account their topology. We want to follow the process of elucidating and analysing the volume of these lattice polyhedra emphasizing its strong relation with some topological features.

To compute the volume of a figure, the more universal method is to cover the figure by units of volume. More explicitly, for each positive real number δ we denote by L_δ the *lattice* in the Euclidean space $\mathbb{R}^n$ consisting of all points of the form $(\delta a_1, \ldots, \delta a_n)$, where $a_1, \ldots, a_n$ are integers. A *unit cube* (*hypercube*) will be any closed cube of volume δ^n whose vertices belong to L_δ. Now, to compute the volume of a figure in $\mathbb{R}^n$, we count the number of unit cubes of L_δ contained in the figure (inner volume) and the number of unit cubes of L_δ whose intersection with the figure is non-empty (exterior volume). If we are interested in a better approximation, we simply decrease δ. By iterating the process, when the limits of the inner volume and the exterior volume exist and coincide, we obtain the desired volume. Thus, to compute volumes, the polyhedra obtained by glueing unit cubes of a lattice play a main role.

In what follows, L will denote the *fundamental lattice* of $\mathbb{R}^n$, consisting of all points of $\mathbb{R}^n$ with integer coordinates, i.e. the lattice L_1. An *L-polyhedron* (*L-hyperpolyhedron*) will be the connected union of many unit cubes of L. The volume of an L-polyhedron P can be obtained by counting the number of unit cubes