and

$$L(f^k) = (-1)^k k E_k(A) \neq 0\,.$$

Therefore, from Theorem 3, f^k has a fixed point and Theorem 2 is proved.

Now we give examples of two continuous self-maps f and g of a connected finite graph G in the hypotheses of Theorem 2 with $k = 2$ such that f has a fixed point and has no periodic points of period 2, and g has no fixed points and has periodic points of period 2. Let G be

EXTENDING THE *-PRODUCT OPERATOR

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Abstract. The *-product operator is a useful tool to describe period doubling or periodic structure for unimodal maps. We define an analogous *-product for other kind of maps as Lorenz and bimodal maps. By using it we show that to each kneading pair with at least one finite component corresponds a box. For bimodal maps we obtain five kind of different geometric boxes. These play a leading role in the description of the structure of the symbolic space of kneading pairs and they can be used to describe the behaviour of topological entropy on it.

1. UNIMODAL MAPS.

Consider a continuous map f of the interval I=[0,1] into itself such that f(0)=f(1)=0 and which has some point $x_C \in (0,1)$ which divides I into two subintervals $[0, x_C]$ and $[x_C, 1]$ in such a way that f is non-decreasing in $[0, x_C]$ and non-increasing in $[x_C, 1]$. These maps will be called *unimodal maps.* Define *A(x)*, for any point x in the interval, by the formal symbol 1, C or 2 according to x belongs to $[0, x_C)$, $\{x_C\}$ or $(x_C, 1]$ respectively. The *itinerary* of a point x, denoted $\underline{\text{It}}(x)$, is the sequence A(x), A(f(x)), $A(f^2(x))$,..., which will be an infinite sequence of 1's and 2's or a finite sequence of 1's and 2's followed by C. To every map f corresponds its *kneading sequence* $\underline{\text{It}}(x_C)$. We shall say that a finite symbolic sequence of 1's and 2's is *even* or *odd* if its number of 2's is even or odd respectively.

It is well known (see [CE] or [DGP] for example) that we can define the **-product* between symbolic sequences $\underline{A}C = A_1 \cdots A_n C$ and $\underline{B} = B_1 \cdots B_n \cdots$ in the following way

$$\underline{A}C * \underline{B} = \underline{A}\, w(B_1)\, \underline{A}\, w(B_2) \ldots$$

where $w(B_i) = B_i$ if $\underline{A}$ is even or $w(B_i) = k(B_i)$ if $\underline{A}$ is odd, where k(1)=2, k(C)=C and k(2) = 1. This *-product can be used to obtain the kneading sequences of maps related by period doubling (see [MSS]). In fact, if $\underline{A}C$ is the kneading sequence of a 2^n superstable orbit we obtain the kneading sequence of a 2^{n+1} superstable orbit as $\underline{A}C*2C$. We can use first-return maps as a basis for the renormalisation scheme (see [CE] or [PTT]). These first-return maps are obtained as an iterated of a unimodal map f restricted to some subinterval J of I. Then the relation between the kneading