LIMIT CYCLES OF 3-DIMENSIONAL DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS FORMED BY LINEAR CENTERS

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Abstract. In this paper we deal with 3-dimensional discontinuous piecewise differential systems formed by linear centers and separated by one plane or two parallel planes. We prove that these systems separated by one plane have no limit cycles, besides the systems separated by two parallel planes have at most one limit cycle, and that there are systems having such a limit cycle. So we solve the extension of the 16th Hilbert problem to this class of differential systems.

1. Introduction and Statement of the Main Results

One of the main goals in the qualitative theory is to study the number of limit cycles of the differential systems. In part this problem was motivated by the 16-th Hilbert problem (1900), see [8, 16] for more details. Limit cycles play a main role for understanding the dynamics of many systems, see for instance [1, 2, 6, 9, 10].

On the other hand there are many problems that are modeled using discontinuous piecewise differential systems. These systems appear in various situations like mechanical systems and control theory, see for instance [3, 4]. In particular the study of *discontinuous piecewise linear differential systems* in the plane separated by straight lines is also an important class of differential systems which appear in models of mechanics and electrical circuits among others, see for instance [5, 13, 15].

Following the Filippov's convention [7] the discontinuous piecewise linear differential systems can have *sliding* or *crossing* limit cycles. A sliding limit cycle contains *sliding segments* on the line of discontinuity, whereas *crossing limit cycles* contain only *crossing points*. In this paper we work with crossing limit cycles, or simply *limit cycles*.

In [11] it was proved that discontinuous piecewise linear differential systems in the plane separated by one straight line and formed by two linear centers have no limit cycles, besides discontinuous piecewise linear differential systems in the plane separated by two parallel straight lines and formed by three linear centers have at most one limit cycle, and that there are systems having such a limit cycle.

The main goal of this paper is to extend these previous results to 3-dimensional discontinuous piecewise differential systems in the space separated by one plane or

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two parallel planes and formed by linear centers. More precisely, we consider the differential system

$$\dot{x} = -y, \quad \dot{y} = x, \quad \dot{z} = 0. \tag{1}$$

This differential system has the $z-$axis filled of singular points, and at every plane $z = z_0$, with z_0 a constant, there exists a linear center at $(0, 0, z_0)$, i.e. all the orbits in this plane are periodic formed by circles centered at $(0, 0, z_0)$, of course with the exception of the singular point $(0, 0, z_0)$.

We shall pass from the coordinates (x, y, z) to (X, Y, Z) through the affine change of coordinates

$$\begin{aligned} X &= a_0 + a_1x + a_2y + a_3z, \\ Y &= b_0 + b_1x + b_2y + b_3z, \\ Z &= c_0 + c_1x + c_2y + c_3z, \end{aligned}$$

assuming that the determinant of matrix of the change of coordinates

$$D = D(a, b, c) = -a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 - a_2b_3c_1 - a_3b_1c_2 + a_3b_2c_1 \neq 0,$$

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $c = (c_1, c_2, c_3)$. In the new coordinates (X, Y, Z) the differential system (1) writes

$$\begin{aligned} \dot{X} &= P_0 + P_1X + P_2Y + P_3Z, \\ \dot{Y} &= Q_0 + Q_1X + Q_2Y + Q_3Z, \\ \dot{Z} &= R_0 + R_1X + R_2Y + R_3Z, \end{aligned} \tag{2}$$

where

$$\begin{aligned}
P_0 &= -D^{-1}(a_1(-a_0b_1c_3 + a_0b_3c_1 - a_3b_0c_1 + a_3b_1c_0) \\
&\quad +a_2(-a_0b_2c_3 + a_0b_3c_2 + a_2b_0c_3 - a_2b_3c_0 - a_3b_0c_2 + a_3b_2c_0) \\
&\quad +a_1^2(b_0c_3 - b_3c_0)), \\
P_1 &= D^{-1}(-a_1b_1c_3 + a_1b_3c_1 - a_2b_2c_3 + a_2b_3c_2), \\
P_2 &= D^{-1}(c_3\left(a_1^2 + a_2^2\right) - a_3(a_1c_1 + a_2c_2)), \\
P_3 &= -D^{-1}(b_3\left(a_1^2 + a_2^2\right) - a_3(a_1b_1 + a_2b_2)), \\
Q_0 &= D^{-1}(c_3\left(a_0\left(b_1^2 + b_2^2\right) - a_1b_0b_1 - a_2b_0b_2\right) + b_3(-a_0(b_1c_1 + b_2c_2) \\
&\quad +a_1b_1c_0 + a_2b_2c_0) + a_3\left(b_0b_1c_1 + b_2(b_0c_2 - b_2c_0) - b_1^2c_0\right)), \\
Q_1 &= -D^{-1}(c_3\left(b_1^2 + b_2^2\right) - b_3(b_1c_1 + b_2c_2)), \\
Q_2 &= D^{-1}(c_3(a_1b_1 + a_2b_2) - a_3(b_1c_1 + b_2c_2)), \\
Q_3 &= -D^{-1}(b_3(a_1b_1 + a_2b_2) - a_3\left(b_1^2 + b_2^2\right)), \\
R_0 &= -D^{-1}(-a_0b_1c_1c_3 - a_0b_2c_2c_3 + a_0b_3c_1^2 + a_0b_3c_2^2 + a_1c_1(b_0c_3 - b_3c_0) \\
&\quad +a_2b_0c_2c_3 - a_2b_3c_0c_2 + a_3\left(-b_0\left(c_1^2 + c_2^2\right) + b_1c_0c_1 + b_2c_0c_2\right)), \\
R_1 &= -D^{-1}(c_3(b_1c_1 + b_2c_2) - b_3\left(c_1^2 + c_2^2\right)), \\
R_2 &= -D^{-1}(a_3\left(c_1^2 + c_2^2\right) - c_3(a_1c_1 + a_2c_2)), \\
R_3 &= D^{-1}(-a_1b_3c_1 - a_2b_3c_2 + a_3b_1c_1 + a_3b_2c_2).
\end{aligned}$$

For simplicity we rename the variables X, Y and Z by x, y and z, respectively. Thus system (2) writes

$$(\dot{x}, \dot{y}, \dot{z})^T = V + M \cdot (x, y, z)^T,$$

where

$$V(a,b,c) = (P_0, Q_0, R_0)^T \quad \text{and} \quad M(a,b,c) = \begin{pmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{pmatrix}.$$

Now we consider the 3-dimensional discontinuous piecewise differential system separated by plane $x = 0$

$$(3) \qquad \begin{array}{ll} (\dot{x}, \dot{y}, \dot{z})^T = V(a,b,c) + M(a,b,c) \cdot (x,y,z)^T, & \text{in } x > 0, \\ (\dot{x}, \dot{y}, \dot{z})^T = V(\alpha,\beta,\gamma) + M(\alpha,\beta,\gamma) \cdot (x,y,z)^T, & \text{in } x < 0, \end{array}$$

and the one separated by the parallel planes $x = -1$ and $x = 1$

$$(4) \qquad \begin{array}{ll} (\dot{x}, \dot{y}, \dot{z})^T = V(a,b,c) + M(a,b,c) \cdot (x,y,z)^T, & \text{in } x > 1, \\ (\dot{x}, \dot{y}, \dot{z})^T = V(A,B,C) + M(A,B,C) \cdot (x,y,z)^T, & \text{in } |x| < 1, \\ (\dot{x}, \dot{y}, \dot{z})^T = V(\alpha,\beta,\gamma) + M(\alpha,\beta,\gamma) \cdot (x,y,z)^T, & \text{in } x < -1, \end{array}$$

where we assume that $D(a,b,c) \neq 0$, $D(\alpha,\beta,\gamma) \neq 0$ and $D(A,B,C) \neq 0$.

In what follows we state the main results of this paper.

Theorem 1. *The discontinuous piecewise linear differential systems* (3) *have no limit cycles.*

Theorem 2. *The discontinuous piecewise linear differential systems* (4) *have at most one limit cycle. Moreover there are systems in this class having one limit cycle.*

Theorems 1 and 2 are proved in sections 2 and 3, respectively. The main tool for proving these theorems is the use of the first integrals of the differential systems which form the discontinuous piecewise differential systems, this technique for studying the limit cycles already was used in [12].

Proposition 1. *The discontinuous piecewise linear differential system separated by the two parallel planes $x = -1$ and $x = 1$*

$$(5) \qquad \begin{array}{llll} \dot{x} = p_0, & \dot{y} = y - z, & \dot{z} = p_0, & \text{in } x > 1, \\ \dot{x} = p_1, & \dot{y} = p_2, & \dot{z} = p_3, & \text{in } |x| < 1, \\ \dot{x} = -x + 2y - 2z, & \dot{y} = -x + 2y - 2z, & \dot{z} = -x + 2y - z, & \text{in } x < -1, \end{array}$$

where

$$p_0 = -\frac{1}{2} - \frac{x}{2} + 2y - \frac{z}{2}, \qquad p_1 = -\frac{13x}{16} + \frac{11y}{8} - \frac{21z}{8} + \frac{5}{16},$$
$$p_2 = \frac{5x}{8} + \frac{y}{4} - \frac{3z}{4} + \frac{3}{8}, \qquad p_3 = \frac{33x}{32} - \frac{7y}{16} + \frac{9z}{16} + \frac{7}{32},$$

has one limit cycle, the one of Figure 1. Moreover this limit cycle is stable.

Proposition 1 is proved at the end of section 3.

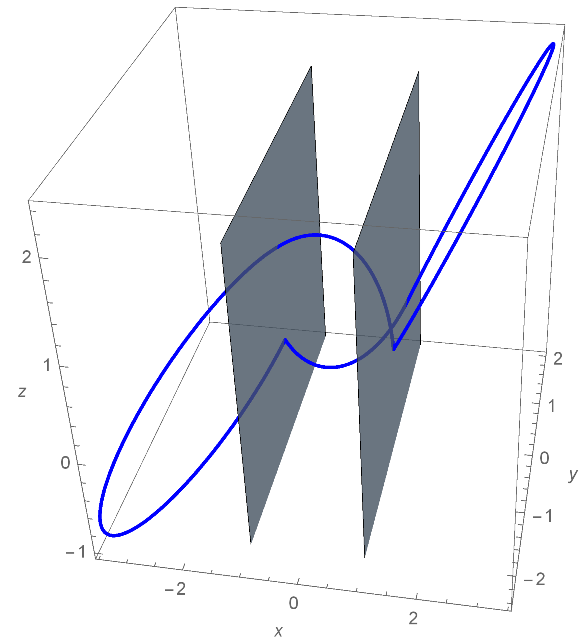

FIGURE 1. The limit cycle of discontinuous piecewise linear differential system (5).

2. PROOF OF THEOREM 1

The differential system in $x > 0$ of the discontinuous linear piecewise differential system (3) has the two independent first integrals

$$\begin{aligned}
h_1(x,y,z) &= (-a_3b_1c_0 + a_1b_3c_0 + a_3b_0c_1 - a_0b_3c_1 - a_1b_0c_3 + a_0b_1c_3 \\
&\quad +b_3c_1x - b_1c_3x - a_3c_1y + a_1c_3y + a_3b_1z - a_1b_3z)^2 \\
&\quad +(a_3b_2c_0 - a_2b_3c_0 - a_3b_0c_2 + a_0b_3c_2 + a_2b_0c_3 - a_0b_2c_3 \\
&\quad -b_3c_2x + b_2c_3x + a_3c_2y - a_2c_3y - a_3b_2z + a_2b_3z)^2, \\
h_2(x,y,z) &= -a_2b_1c_0 + a_1b_2c_0 + a_2b_0c_1 - a_0b_2c_1 - a_1b_0c_2 + a_0b_1c_2 \\
&\quad +(b_2c_1 - b_1c_2)x + (-a_2c_1 + a_1c_2)y + (a_2b_1 - a_1b_2)z,
\end{aligned}$$

and the differential system in $x < 0$ has the two independent first integrals f_1 and f_2 obtained respectively from h_1 and h_2, changing the parameters a_i, b_i and c_i by α_i, β_i and γ_i respectively, for $i = 0, 1, 2, 3$.

A limit cycle for discontinuous piecewise differential system (3) must intersect the plane $x = 0$ in two distinct points, denoted by $(0, y_1, z_1)$ and $(0, y_2, z_2)$, and such two points must satisfy the system of equations

$$\begin{aligned}
e_1 &= h_1(0,y_2,z_2) - h_1(0,y_1,z_1) = 0, \\
e_2 &= h_2(0,y_2,z_2) - h_2(0,y_1,z_1) = 0, \\
e_3 &= f_1(0,y_2,z_2) - f_1(0,y_1,z_1) = 0, \\
e_4 &= f_2(0,y_2,z_2) - f_2(0,y_1,z_1) = 0.
\end{aligned}$$

Taking the change of variables $y_1 = y + y_2$ and $z_1 = z + z_2$ we obtain

$$\begin{aligned} e_2 &= (a_2c_1 - a_1c_2)y + (-a_2b_1 + a_1b_2)z, \\ e_4 &= (\alpha_2\gamma_1 - \alpha_1\gamma_2)y + (-\alpha_2\beta_1 + \alpha_1\beta_2)z. \end{aligned}$$

In order that the system $e_2 = e_4 = 0$ has non-trivial solutions we need that the following determinant be zero

$$\begin{aligned} \Delta = & -a_2\alpha_2\beta_1c_1 + \alpha_1a_2\beta_2c_1 + a_2\alpha_2b_1\gamma_1 - a_1\alpha_2b_2\gamma_1 + a_1\alpha_2\beta_1c_2 \\ & -a_1\alpha_1\beta_2c_2 - \alpha_1a_2b_1\gamma_2 + a_1\alpha_1b_2\gamma_2. \end{aligned}$$

However now there are only three independent equations $e_1 = e_2 = e_3 = 0$ and four unknowns variables y_1, y_2, z_1 and z_2. Thus always we have a continuum of periodic solutions and no limit cycles. Therefore Theorem 1 is proved.

3. Proof of Theorem 2

We note that the differential system in $x > 1$ of the discontinuous piecewise differential system (4) has the same two independent first integrals h_1 and h_2 given in the proof of Theorem 1. The differential system in $|x| < 1$ has the two independent first integrals f_1 and f_2 obtained respectively from h_1 and h_2 replacing the parameters a_i, b_i and c_i by A_i, B_i and C_i, respectively for $i = 0, 1, 2, 3$, and the differential system in $x < -1$ has the two independent first integrals g_1 and g_2 obtained respectively from h_1 and h_2, changing the parameters a_i, b_i and c_i by α_i, β_i and γ_i respectively for $i = 0, 1, 2, 3$.

A limit cycle of the discontinuous piecewise differential system (4) must intersect each plane $x = 1$ and $x = -1$ in two distinct points, denoted by $(1, y_1, z_1)$ and $(1, y_2, z_2)$, and $(-1, y_3, z_3)$ and $(-1, y_4, z_4)$, respectively. Such four points must satisfy the system of equations

(6)

$$\begin{array}{ll} E_1 = h_1(1, y_2, z_2) - h_1(1, y_1, z_1) = 0, & E_2 = h_2(1, y_2, z_2) - h_2(1, y_1, z_1) = 0, \\ E_3 = f_1(1, y_2, z_2) - f_1(-1, y_3, z_3) = 0, & E_4 = f_2(1, y_2, z_2) - f_2(-1, y_3, z_3) = 0, \\ E_5 = g_1(-1, y_3, z_3) - g_1(-1, y_4, z_4) = 0, & E_6 = g_2(-1, y_3, z_3) - g_2(-1, y_4, z_4) = 0, \\ E_7 = f_1(-1, y_4, z_4) - f_1(1, y_1, z_1) = 0, & E_8 = f_2(-1, y_4, z_4) - f_2(1, y_1, z_1) = 0. \end{array}$$

Applying the change of parameters

$$\begin{array}{lllllllll} K_0 &=& b_3c_1 - b_1c_3, & K_1 &=& a_3c_1 - a_1c_3, & K_2 &=& a_3b_1 - a_1b_3, \\ K_3 &=& b_3c_2 - b_2c_3, & K_4 &=& a_3c_2 - a_2c_3, & K_5 &=& a_3b_2 - a_2b_3, \\ K_6 &=& a_2c_1 - a_1c_2, & K_7 &=& a_2b_1 - a_1b_2, & K_8 &=& B_3C_1 - B_1C_3, \\ K_9 &=& A_3C_1 - A_1C_3, & K_{10} &=& A_3B_1 - A_1B_3, & K_{11} &=& B_3C_2 - B_2C_3, \\ K_{12} &=& A_3C_2 - A_2C_3, & K_{13} &=& A_3B_2 - A_2B_3, & K_{14} &=& B_2C_1 - B_1C_2, \\ K_{15} &=& A_2C_1 - A_1C_2, & K_{16} &=& A_2B_1 - A_1B_2, & K_{17} &=& \alpha_3\beta_1 - \alpha_1\beta, \\ K_{18} &=& \alpha_3\gamma_1 - \alpha_1\gamma_3, & K_{19} &=& \alpha_3\beta_2 - \alpha_2\beta_3, & K_{20} &=& \beta_1\gamma_3 - \beta_3\gamma_1, \\ K_{21} &=& \alpha_3\gamma_2 - \alpha_2\gamma_3, & K_{22} &=& \beta_2\gamma_3 - \beta_3\gamma_2, & K_{23} &=& \alpha_2\beta_1 - \alpha_1\beta_2, \\ K_{24} &=& \alpha_2\gamma_1 - \alpha_1\gamma_2, & & & & & & \end{array}$$

the polynomials E_i for $i = 1, ..., 8$ write

$$\begin{aligned}
E_1 &= -((a_0-1)K_0 + K_1(y_1-b_0) + K_2(c_0-z_1))^2 + ((a_0-1)K_0 \\
&\quad +K_1(y_2-b_0) + K_2(c_0-z_2))^2 - ((a_0-1)K_3 + K_4(y_1-b_0) \\
&\quad +K_5(c_0-z_1))^2 + ((a_0-1)K_3 + K_4(y_2-b_0) + K_5(c_0-z_2))^2, \\
E_2 &= K_6(y_1-y_2) - K_7(z_1-z_2), \\
E_3 &= ((A_0-1)K_8 + K_9(y_2-B_0) + K_{10}(C_0-z_2))^2 - ((A_0+1)K_8 \\
&\quad +K_9(y_3-B_0) + K_{10}(C_0-z_3))^2 + ((A_0-1)K_{11} + K_{12}(y_2-B_0) \\
&\quad +K_{13}(C_0-z_2))^2 - ((A_0+1)K_{11} + K_{12}(y_3-B_0) + K_{13}(C_0-z_3))^2, \\
E_4 &= 2K_{14} - K_{15}(y_2-y_3) + K_{16}(z_2-z_3), \\
E_5 &= (K_{17}(z_3-\gamma_0) + K_{18}(\beta_0-y_3) + (\alpha_0+1)K_{20})^2 - (K_{17}(z_4-\gamma_0) \\
&\quad +K_{18}(\beta_0-y_4) + (\alpha_0+1)K_{20})^2 + (K_{19}(z_3-\gamma_0) + K_{21}(\beta_0-y_3) \\
&\quad +(\alpha_0+1)K_{22})^2 - (K_{19}(z_4-\gamma_0) + K_{21}(\beta_0-y_4) + (\alpha_0+1)K_{22})^2, \\
E_6 &= K_{23}(z_3-z_4) - K_{24}(y_3-y_4), \\
E_7 &= -((A_0-1)K_8 + K_9(y_1-B_0) + K_{10}(C_0-z_1))^2 + ((A_0+1)K_8 \\
&\quad +K_9(y_4-B_0) + K_{10}(C_0-z_4))^2 - ((A_0-1)K_{11} + K_{12}(y_1-B_0) \\
&\quad +K_{13}(C_0-z_1))^2 + ((A_0+1)K_{11} + K_{12}(y_4-B_0) + K_{13}(C_0-z_4))^2, \\
E_8 &= -2K_{14} + K_{15}(y_1-y_4) - K_{16}(z_1-z_4).
\end{aligned}$$

Remark 1. If $(y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4)$ is a solution of system (6), then $(y_2, y_1, y_4, y_3, z_2, z_1, z_4, z_3)$ is also a solution.

In what follows we provide 14 lemmas.

Lemma 1. *If $K_6 \neq 0$, $K_{24} \neq 0$, $K_{15} \neq 0$ and $K_{16}K_6 - K_{15}K_7 \neq 0$, then system (4) has at most one limit cycle.*

Proof. Since $K_6 \neq 0$ and $K_{24} \neq 0$, we solve $E_2 = 0$ in the variable y_1 and $E_6 = 0$ in the variable y_3, and we obtain

$$y_1 = y_2 + \frac{K_7}{K_6}(z_1 - z_2), \tag{7}$$

$$y_3 = y_4 + \frac{K_{23}}{K_{24}}(z_3 - z_4). \tag{8}$$

Replacing y_1 and y_3 in each equation $E_i = 0$ for $i \in \{1, 3, 4, 5, 7, 8\}$, in particular we obtain that

$$E_4 = 2K_{14} + K_{16}(z_2 - z_3) + K_{15}(y_4 - y_2) + \frac{K_{15}K_{23}}{K_{24}}(z_3 - z_4) = 0. \tag{9}$$

Since $K_{15} \neq 0$ we solve $E_4 = 0$ in the variable y_2 and we substitute it in each $E_i = 0$ for $i \in \{1, 3, 5, 7, 8\}$. Thus E_8 becomes

$$E_8 = K_{16}(-z_1 + z_2 - z_3 + z_4) + \frac{K_7K_{15}}{K_6}(z_1 - z_2) + \frac{K_{15}K_{23}}{K_{24}}(z_3 - z_4) = 0.$$

Since $K_{16}K_6 - K_{15}K_7 \neq 0$ we solve $E_8 = 0$ in the variable z_1 and we substitute it in each equation $E_i = 0$ for $i \in \{1, 3, 5, 7\}$. Now we have the following four polynomial equations in the variables z_2, z_3, z_4 and y_4:

$$\begin{aligned}
E_1 &= \frac{(K_{15}K_{23} - K_{16}K_{24})}{K_{15}K_{24}^2(K_{16}K_6 - K_{15}K_7)^2}(z_3 - z_4)T_1 = 0,\\
E_3 &= -\frac{1}{K_{15}^2K_{24}}T_2 = 0,\\
E_5 &= -\frac{1}{K_{24}^2}(z_3 - z_4)T_3 = 0,\\
E_7 &= \frac{1}{K_{15}^2K_{24}^2(K_{16}K_6 - K_{15}K_7)^2}T_4 = 0,
\end{aligned}$$

where T_1 and T_3 are polynomials of degree 1 and T_2 and T_4 are polynomials of degree 2. We do not explicit them here due to their lengths. If $z_3 = z_4$ then from (8) we get $y_3 = y_4$ and we have no limit cycles. Thus we must study the zeros of the system $T_1 = T_2 = T_3 = T_4 = 0$. Taking $E_{3,7} = E_3 + E_7$ we get the new equation

$$E_{3,7} = \frac{1}{K_{15}K_{24}^2(K_{16}K_6 - K_{15}K_7)^2}(z_3 - z_4)T_5 = 0,$$

such that T_5 is a polynomial of degree 1. In short, we have four polynomials and the product of their degrees is 2.

Thus if system (6) has finitely many solutions by Bezout Theorem (see for instance [14]) it has at most 2 solutions. By Remark 1 these two solutions correspond to the same limit cycle. So the discontinuous piecewise linear differential system (4) has at most one limit cycle. □

Lemma 2. *If $K_6 \neq 0$, $K_{24} = 0$, $K_{15} \neq 0$ and $K_{16}K_6 - K_{15}K_7 \neq 0$, then system* (4) *has at most one limit cycle.*

Proof. Since $K_6 \neq 0$ we solve $E_2 = 0$ in the variable y_1 and we obtain (7). Taking $K_{24} = 0$ and replacing y_1 given in (7) in each $E_i = 0$, for $i \in \{1, 3, .., 8\}$, in particular we get $E_6 = K_{23}(z_3 - z_4) = 0$. So $z_3 = z_4$, otherwise $K_{23} = 0$ and we do not have finitely many solutions and therefore we have no limit cycles. Then we have that

$$E_8 = -2K_{14} + K_{16}z_4 + K_{15}(y_2 - y_4) - \frac{1}{K_6}\Big((K_{16}K_6 - K_{15}K_7)z_1 + K_{15}K_7z_2\Big).$$

Solving $E_8 = 0$ in the variable y_2 we obtain

$$y_2 = y_4 + \frac{2K_{14}K_6 + K_{16}K_6z_1 - K_{15}K_7z_1 + K_{15}K_7z_2 - K_{16}K_6z_4}{K_{15}K_6}.$$

Substituting y_2 in each $E_i = 0$ for $i \in \{1, 3, 4, 5, 7\}$, in particular we have

$$E_4 = K_{15}(y_3 - y_4) - \frac{1}{K_6}(K_{16}K_6 - K_{15}K_7)(z_1 - z_2) = 0.$$

Solving $E_4 = 0$ in the variable y_3 and substituting it in each $E_i = 0$ for $i \in \{1, 3, 5, 7\}$ we obtain the following four polynomial equations in the variables z_1, z_2, y_4 and z_4

$$\begin{aligned}
E_1 &= -\frac{1}{K_{15}K_6^2}(z_2 - z_1)T_1 = 0,\\
E_3 &= -\frac{1}{K_{15}^2 K_6}T_2 = 0,\\
E_5 &= \frac{(K_{16}K_6 - K_{15}K_7)}{K_{15}^2 K_6^2}(z_2 - z_1)T_3 = 0,\\
E_7 &= \frac{1}{K_{15}^2}T_4 = 0,
\end{aligned}$$

where T_1 and T_3 are polynomials of degree 1 and T_2 and T_4 are polynomials of degree 2. If $z_1 = z_2$ then from (7) we have $y_1 = y_2$ and we have no limit cycles. Thus we must study the number of the zeros of the system $T_1 = T_2 = T_3 = T_4 = 0$. Taking $E_{3,7} = E_3 + E_7$ we get the new equation

$$E_{3,7} = \frac{1}{K_{15}^2 K_6}(z_1 - z_2)T_5 = 0,$$

such that T_5 is a polynomial of degree 1. In short we have four polynomials and the product of their degrees is 2. Thus as in the proof of the previous lemma we conclude that the discontinuous piecewise differential system (4) has at most one limit cycle. □

Lemma 3. *If $K_6 \neq 0$, $K_{24} \neq 0$, $K_{15} = 0$ and $K_{16}K_6 - K_{15}K_7 \neq 0$, then system* (4) *has at most one limit cycle.*

Proof. Consider $K_{15} = 0$. So $K_{16} \neq 0$. Since $K_6 \neq 0$ and $K_{24} \neq 0$ we substituted y_1 and y_3 given in (7) and (8) respectively, in each equation $E_i = 0$ for $i = 1, ..., 8$. Thus E_4 and E_8 become respectively

$$E_4 = 2K_{14} + K_{16}z_2 - K_{16}z_3, \quad E_8 = -2K_{14} - K_{16}z_1 + K_{16}z_4.$$

Solving $E_4 = 0$ and $E_8 = 0$ in the variables z_2 and z_1 respectively, we obtain

$$z_1 = (-2K_{14} + K_{16}z_4)/K_{16}, \quad z_2 = (-2K_{14} + K_{16}z_3)/K_{16}.$$

Substituting them in each equation $E_i = 0$ for $i \in \{1, 3, 5, 7\}$, we obtain the following four polynomial equations in the variables y_2, y_3, z_3 and z_4:

$$\begin{aligned}
E_1 &= \frac{1}{K_{16}K_6^2}(z_3 - z_4)T_1 = 0,\\
E_3 &= -\frac{1}{K_{16}^2 K_{24}^2}T_2 = 0,\\
E_5 &= -\frac{1}{K_{24}^2}(z_3 - z_4)T_3 = 0,\\
E_7 &= \frac{1}{K_{16}^2 K_6^2}T_4 = 0,
\end{aligned}$$

where T_1 and T_3 are polynomials of degree 1 and T_2 and T_4 are polynomials of degree 2. As in the proof of Lemma 1 we conclude that the discontinuous piecewise differential system (4) has at most one limit cycle. □

Lemma 4. *If $K_6 = 0$, $K_{24} \neq 0$, $K_{15} \neq 0$ and $K_{16}K_6 - K_{15}K_7 \neq 0$, then system* (4) *has at most one limit cycle.*

Proof. Consider $K_6 = 0$. So we have $E_2 = -K_7(z_1 - z_2) = 0$. Since $K_7 \neq 0$, we have that $z_1 = z_2$. Taking $z_1 = z_2$ and solving $E_6 = 0$ in the variable y_3 we obtain (8). Substituting it in each equation $E_i = 0$ for $i \in \{1, 3, 4, 5, 7, 8\}$, in particular we obtain $E_4 = 0$ given in (9). Solving $E_4 = 0$ in the variable y_2 and substituting it in each equation $E_i = 0$ for $i \in \{1, 3, 5, 7, 8\}$, E_8 becomes

$$E_8 = -2K_{14} + K_{15}(y_1 - y_4) + K_{16}(z_4 - z_2).$$

Solving $E_8 = 0$ in the variable y_1 and substituting it in each equation $E_i = 0$ for $i \in \{1, 3, 5, 7\}$, we obtain the following four polynomial equations in the variables y_4, z_2, z_3 and z_4:

$$\begin{aligned}
E_1 &= \frac{K_{15}K_{23} - K_{16}K_{24}}{K_{15}^2K_{24}^2}(z_3 - z_4)T_1 = 0,\\
E_3 &= -\frac{1}{K_{15}^2K_{24}}T_2 = 0,\\
E_5 &= -\frac{1}{K_{24}^2}(z_3 - z_4)T_3 = 0,\\
E_7 &= \frac{1}{K_{15}^2}T_4 = 0,
\end{aligned}$$

where T_1 and T_3 are polynomials of degree 1 and T_2 and T_4 are polynomials of degree 2. As in the proof of Lemma 1 we conclude that the discontinuous piecewise differential system (4) has at most one limit cycle. □

Lemma 5. *If $K_6 = 0$, $K_{24} = 0$, $K_{15} \neq 0$ and $K_{16}K_6 - K_{15}K_7 \neq 0$, then system* (4) *has at most one limit cycle.*

Proof. Consider $K_6 = 0$, $K_{24} = 0$ and $K_{15} \neq 0$. So we have that $K_7 \neq 0$. From $E_2 = 0$ and $E_6 = 0$ we obtain that $z_1 = z_2$ and $z_3 = z_4$. Solving $E_4 = 0$ and $E_8 = 0$ in the variables y_2 and y_1, respectively and substituting them in each $E_i = 0$ for $i \in \{1, 3, 5, 7\}$ we get the following four polynomial equations in the variables y_3, y_4, z_2 and z_4:

$$\begin{aligned}
E_1 &= \frac{1}{K_{15}}(y_3 - y_4)T_1 = 0,\\
E_3 &= -\frac{1}{K_{15}^2}T_2 = 0,\\
E_5 &= (y_4 - y_3)T_3 = 0,\\
E_7 &= \frac{1}{K_{15}^2}T_4 = 0,
\end{aligned}$$

where T_1 and T_3 are polynomials of degree 1 and T_2 and T_4 are polynomials of degree 2. Again as in the proof of Lemma 1 we conclude that the discontinuous piecewise differential system (4) has at most one limit cycle. □

Lemma 6. *If $K_6 \neq 0$, $K_{24} = 0$, $K_{15} = 0$ and $K_{16}K_6 - K_{15}K_7 \neq 0$, then system* (4) *has no limit cycles.*

Proof. Take $K_{24} = 0$ and $K_{15} = 0$. So we have that $K_{16} \neq 0$, $E_6 = K_{23}(z_3 - z_4)$ and $E_8 = -2K_{14} - K_{16}z_1 + K_{16}z_4$. As in the proof of Lemma 2 we have that $z_3 = z_4$. Solving $E_2 = 0$ in the variable y_1 we obtain (7). Solving $E_8 = 0$ in the variable z_4 and substituting it in $E_4 = 0$ we get $E_4 = -K_{16}(z_1 - z_2) = 0$. If $z_1 = z_2$ we have from (7) that $y_1 = y_2$ and we have no limit cycles. Therefore system (4) has no limit cycles. □

Lemma 7. *If $K_6 \neq 0$, $K_{24} \neq 0$, $K_{15} \neq 0$ and $K_{16}K_6 - K_{15}K_7 = 0$, then system* (4) *has no limit cycles.*

Proof. Consider $K_6 \neq 0$, $K_{24} \neq 0$, $K_{15} \neq 0$ and $K_{16}K_6 - K_{15}K_7 = 0$. Solving $E_2 = 0$ and $E_6 = 0$ in the variables y_1 and y_3 respectively, we obtain y_1 and y_3 given in (7) and (8). Substituting y_1, y_3 and $K_{16} = K_{15}K_7/K_6$ in $E_4 = 0$ and $E_8 = 0$ in particular we obtain

$$E_4 = 2K_{14} + K_{15}(y_4 - y_2) + \frac{K_{15}K_7}{K_6}(z_2 - z_3) + \frac{K_{15}K_{23}}{K_{24}}(z_3 - z_4) = 0.$$

Solving $E_4 = 0$ in the variable y_2 and substituting it in $E_8 = 0$ we obtain

$$E_8 = \frac{K_{15}(K_{23}K_6 - K_{24}K_7)(z_3 - z_4)}{K_{24}K_6} = 0.$$

If $z_3 = z_4$, then from (8) we get $y_3 = y_4$ and we have no limit cycles. If $K_{23}K_6 - K_{24}K_7 = 0$, then there are more unknown variables than equations in system (6) and therefore there are no limit cycles. □

Lemma 8. *If $K_6 \neq 0$ and $K_{24} \neq 0$, $K_{15} = 0$ and $K_{16}K_6 - K_{15}K_7 = 0$, then system* (4) *has no limit cycles.*

Proof. Consider $K_6 \neq 0$ and $K_{24} \neq 0$. Taking $K_{15} = 0$ we have that $K_{16} = 0$. So we obtain $E_4 = -E_8 = 2K_{14}$. Therefore there are more unknown variables than equations in system (6) and therefore there are no limit cycles. □

Lemma 9. *If $K_6 \neq 0$ and $K_{24} = 0$, $K_{15} = 0$ and $K_{16}K_6 - K_{15}K_7 = 0$, then system* (4) *has no limit cycles.*

Proof. The proof of this lemma is analogous to the one of the previous lemma. □

Lemma 10. *If $K_6 = 0$, $K_{24} = 0$ and $K_{15} = 0$, then system* (4) *has no limit cycles.*

Proof. Considering $K_6 = 0$, $K_{24} = 0$ and $K_{15} = 0$ we have $E_2 = -K_7(z_1 - z_2)$ and $E_6 = K_{23}(z_3 - z_4)$. So for having limit cycles it is necessary that $z_1 = z_2$ and $z_3 = z_4$. However in this case $E_4 = -E_8 = 2K_{14} + K_{16}z_2 - K_{16}z_4$ and therefore there are no limit cycles. □

Lemma 11. *If $K_6 \neq 0$, $K_{24} = 0$, $K_{15} \neq 0$ and $K_{16}K_6 - K_{15}K_7 = 0$, then system* (4) *has no limit cycles.*

Proof. Consider $K_6 \neq 0$, $K_{24} = 0$, $K_{15} \neq 0$ and $K_{16} = K_{15}K_7/K_6$. Thus we have that $E_6 = K_{23}(z_3 - z_4) = 0$. It is necessary that $z_3 = z_4$ for having limit cycles.

Solving $E_2 = 0$ in the variable y_1 we get y_1 given in (7). Substituting it in $E_4 = 0$ and $E_8 = 0$ we obtain

$$E_4 = \frac{2K_{14}K_6 - K_{15}K_6y_2 + K_{15}K_6y_3 + K_{15}K_7z_2 - K_{15}K_7z_4}{K_6} = 0,$$
$$E_8 = -\frac{2K_{14}K_6 - K_{15}K_6y_2 + K_{15}K_6y_4 + K_{15}K_7z_2 - K_{15}K_7z_4}{K_6} = 0.$$

Solving $E_8 = 0$ in the variable y_2 and substituting it in $E_4 = 0$ we have that $E_4 = K_{15}(y_3 - y_4) = 0$. If $y_3 = y_4$ we have no limit cycles because $z_3 = z_4$. □

Lemma 12. *If $K_6 = 0$, $K_{24} = 0$, $K_{15} \neq 0$ and $K_{16}K_6 - K_{15}K_7 = 0$, then system* (4) *has no limit cycles.*

Proof. Consider $K_6 = 0$, $K_{24} = 0$ and $K_{15} \neq 0$. Since $K_{16}K_6 - K_{15}K_7 = 0$ we have $K_7 = 0$. Thus we vanish E_2 and system (6) has seven equations and eight unknown variables $y_1, .., y_4, z_1, ..., z_4$. So there are no limit cycles. □

Lemma 13. *If $K_6 = 0$, $K_{24} \neq 0$, $K_{15} \neq 0$ and $K_{16}K_6 - K_{15}K_7 = 0$, then system* (4) *has no limit cycles.*

Proof. The proof of this lemma is analogous to the one of the previous lemma. □

Lemma 14. *If $K_6 = 0$, $K_{24} \neq 0$ and $K_{15} = 0$, then system* (4) *has no limit cycles.*

Proof. Consider $K_6 = 0$, $K_{24} \neq 0$ and $K_{15} = 0$. We have that $E_2 = -K_7(z_1 - z_2) = 0$. So for having limit cycles it is necessary that $z_1 = z_2$. Thus we obtain

$$E_8 = -2K_{14} - K_{16}z_2 + K_{16}z_4 = 0.$$

Note that if $K_{16} = 0$ we have no limit cycles. Solving $E_6 = 0$ and $E_8 = 0$ in the variables y_3 and z_2, respectively and substituting them in $E_4 = 0$ and $E_5 = 0$ in particular we obtain $E_4 = -K_{16}(z_3 - z_4) = 0$. Taking $z_3 = z_4$ we vanish E_4 and E_5. So there are no limit cycles because system (6) has more unknown variables than equations. □

Lemmas 1–14 show that if $K_{16}K_6 - K_{15}K_7 = 0$, then the discontinuous piecewise differential system (4) has no limit cycles; and if $K_{16}K_6 - K_{15}K_7 \neq 0$, then it has at most one limit cycle, except when $K_6 \neq 0$, $K_{24} = 0$ and $K_{15} = 0$ (in this case there are no limit cycles).

In short, we conclude that the discontinuous piecewise differential system (4) has at most one limit cycle. In order to complete the proof of Theorem 2 we must prove Proposition 1.

Proof of Proposition 1. We shall prove that discontinuous piecewise linear differential system (5) has one limit cycle.

The first integrals f_1, f_2, g_1, g_2, h_1 and h_2 for the discontinuous linear piecewise differential system (5) are

$$\begin{aligned}
f_1 &= \frac{2}{9}(5+6x+53x^2+16y-80xy+52y^2-36z+132xz-192yz+180z^2),\\
f_2 &= \frac{1}{6}(-5-3x+6y-6z),\\
g_1 &= (-x+2y-z)^2+z^2,\\
g_2 &= -x+y,\\
h_1 &= (-1-x+2y+z)^2+(-2y+2z)^2.\\
h_2 &= -1-x+z.
\end{aligned}$$

So system (6) becomes

$$\begin{aligned}
0 = &\ -8y_1^2+8y_2^2+4z_1-5z_1^2+4y_1(2+z_1)-4z_2+5z_2^2-4y_2(2+z_2),\\
0 = &\ -z_1+z_2,\\
0 = &\ \frac{8}{9}(3+13y_2^2-13y_3^2+24z_2+45z_2^2-16y_2(1+3z_2)+42z_3-45z_3^2,\\
&\ +24y_3(-1+2z_3)),\\
0 = &\ -1+y_2-y_3-z_2+z_3,\\
0 = &\ z_3^2+(-1-2y_3+z_3)^2-z_4^2-(-1-2y_4+z_4)^2,\\
0 = &\ y_3-y_4,\\
0 = &\ -\frac{8}{9}(3+13y_1^2-13y_4^2+24z_1+45z_1^2-16y_1(1+3z_1)+42z_4-45z_4^2,\\
&\ +24y_4(-1+2z_4)),\\
0 = &\ 1-y_1+y_4+z_1-z_4.
\end{aligned}$$

The unique isolated solution of the previous system satisfying $(y_1, z_1) \neq (y_2, z_2)$ and $(y_3, z_3) \neq (y_4, z_4)$ is

$$(y_1^*, y_2^*, y_3^*, y_4^*, z_1^*, z_2^*, z_3^*, z_4^*) = (1, 0, 0, 0, 0, 0, 1, 0).$$

The solution $(x_1(t), y_1(t), z_1(t))$ of system (5) in $x > 1$ such that $(x_1(0), y_1(0), z_1(0)) = (1, y_1^*, z_1^*)$ is

$$\begin{aligned}
x(t) &= 2-\cos t+\sin t,\\
y(t) &= 1+\sin t,\\
z(t) &= 1-\cos t+\sin t.
\end{aligned}$$

The solution $(x_2(t), y_2(t), z_2(t))$ of system (5) in $|x| < 1$ such that $(x_2(0), y_2(0), z_2(0)) = (1, y_2^*, z_2^*)$ is

$$\begin{aligned}
x(t) &= \frac{1}{2}(-1+3\cos t-\sin t),\\
y(t) &= -1+\cos t+\sin t,\\
z(t) &= \frac{1}{4}(-1+\cos t+5\sin t).
\end{aligned}$$

The solution $(x_3(t), y_3(t), z_3(t))$ of system (5) in $x < -1$ such that $(x_3(0), y_3(0), z_3(0)) = (-1, y_3^*, z_3^*)$ is

$$\begin{aligned} x(t) &= -2 + \cos t - \sin t, \\ y(t) &= -1 + \cos t - \sin t, \\ z(t) &= \cos t. \end{aligned}$$

Finally the solution $(x_4(t), y_4(t), z_4(t))$ of system of (5) in $|x| < 1$ such that $(x_4(0), y_4(0), z_4(0)) = (-1, y_4^*, z_4^*)$ is

$$\begin{aligned} x(t) &= \frac{1}{8}(-1 - 7\cos t + 9\sin t), \\ y(t) &= \frac{1}{4}(5 - 5\cos t - \sin t), \\ z(t) &= -\frac{13}{16}(-1 + \cos t + \sin t). \end{aligned}$$

The time that the solution $(x_1(t), y_1(t), z_1(t))$ contained in $x > 1$ needs to reach the point $(1, y_2^*, z_2^*)$ is $t_1 = 3\pi/2$. The time that the solution $(x_2(t), y_2(t), z_2(t))$ contained in $-1 < x < 1$ needs to reach the point $(-1, y_3^*, z_3^*)$ is $t_2 = \pi/2$. The time that the solution $(x_3(t), y_3(t), z_3(t))$ contained in $x < -1$ needs to reach the point $(-1, y_4^*, z_4^*)$ is $t_3 = 3\pi/2$. Lastly the time that the solution $(x_4(t), y_4(t), z_4(t))$ contained in $-1 < x < 1$ needs to reach the point $(1, y_1^*, z_1^*)$ is $t_4 = \pi/2$. The limit cycle of Figure 1 is obtained drawing the orbits $(x_k(t), y_k(t), z_k(t))$, for $t \in [0, t_k]$ and $k = 1, 2, 3, 4$.

Now we shall prove that this limit cycle is stable. This limit cycle starts at the point $(1, 0, 0)$ of the plane $x = 1$, cross the region $|x| < 1$ until the point $(-1, 0, 1)$ of the plane $x = -1$, after travels in the region $x < 1$ until the point $(-1, 0, 0)$ of the plane $x = -1$, cross again the region $|x| < 1$ until the point $(1, 1, 0)$ of the plane $x = 1$, and finally it travels in the region $x > 1$ until the initial point $(1, 0, 0)$.

Let ε and δ be two small real numbers, then the point $(1, \varepsilon, \delta)$ is close to the point $(1, 0, 0)$ of the limit cycle. Using the first integrals f_1 and f_2 we compute where the orbit through the point $(1, \varepsilon, \delta)$ intersect the plane $x = -1$ near the point $(-1, 0, 1)$, this intersection takes place at the point $(-1, y_1, z_1)$ where

$$\begin{aligned} y_1 &= \frac{1}{10}\left(\sqrt{441\delta^2 + 66\delta(4 - 7\varepsilon) + 121\varepsilon^2 - 184\varepsilon + 144} - 21\delta + 21\varepsilon - 12\right), \\ z_1 &= \frac{1}{10}\left(\sqrt{441\delta^2 + 66\delta(4 - 7\varepsilon) + 121\varepsilon^2 - 184\varepsilon + 144} - 11\delta + 11\varepsilon - 2\right). \end{aligned}$$

Now with the first integrals g_1 and g_2 we compute where the orbit through the point $(-1, y_1, z_1)$ travels in the region $x < -1$ until intersecting the plane $x = -1$ at the point $(-1, y_2, z_2)$ near the point $(-1, 0, 0)$, where

$$\begin{aligned} y_2 &= \frac{1}{10}\left(\sqrt{441\delta^2 + 66\delta(4 - 7\varepsilon) + \varepsilon(121\varepsilon - 184) + 144} - 21\delta + 21\varepsilon - 12\right), \\ z_2 &= \frac{1}{10}\left(\sqrt{441\delta^2 + 66\delta(4 - 7\varepsilon) + \varepsilon(121\varepsilon - 184) + 144} - 31\delta + 31\varepsilon - 12\right). \end{aligned}$$

Using again the first integrals f_1 and f_2 we compute where the orbit through the point $(-1, y_2, z_2)$ travels in the region $|x| < 1$ until intersecting the plane $x = 1$ at the point $(1, y_3, z_3)$ near the point $(1, 1, 0)$, where

$$y_3 = \frac{1}{10}\left(21\delta - T - 21\varepsilon + 17\right), \qquad z_3 = \frac{1}{10}\left(11\delta - T - 11\varepsilon + 7\right),$$

here

$$T = \sqrt{2205\delta^2 + 6\delta(354 - 665\varepsilon) - 42R(2\delta - 2\varepsilon + 1) + \varepsilon(1885\varepsilon - 2044) + 553},$$
$$R = \sqrt{441\delta^2 + 66\delta(4 - 7\varepsilon) + \varepsilon(121\varepsilon - 184) + 144}.$$

Finally with the first integrals h_1 and h_2 we compute where the orbit through the point $(1, y_3, z_3)$ travels in the region $x > 1$ until intersecting the plane $x = 1$ at the point $(1, y_4, z_4)$ near the point $(1, 0, 0)$, where

$$y_4 = \frac{1}{20}(-31\delta + S + 31\varepsilon - 7), \qquad z_4 = \frac{1}{10}(11\delta - S - 11\varepsilon + 7),$$

where $S = \sqrt{2205\delta^2 + 6\delta(354 - 665\varepsilon) - 42R(2\delta - 2\varepsilon + 1) + \varepsilon(1885\varepsilon - 2044) + 553}$.

In summary, the Poincaré map F near the point $(1, 0, 0)$ and the limit cycle is $f(1, \varepsilon, \delta) = (1, y_4, z_4)$. Therefore

$$Df(1, 0, 0) = \begin{pmatrix} -1 & \dfrac{11}{14} \\ 4 & -\dfrac{25}{7} \end{pmatrix},$$

and their eigenvalues are $\dfrac{1}{7}\left(-16 \pm \sqrt{235}\right)$, both negative. Hence the limit cycles is stable. This completes the proof of the proposition. ☐

Remark 2. *We note that for the piecewise differential system of Proposition 1 we have that $K_6 = 0$, $K_7 = 1$, $K_{15} = K_{16} = -2$ and $K_{24} = 1$. So this piecewise differential system only satisfies* Lemma 4.

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