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Pseudo-Anosov homeomorphisms on a sphere with four punctures have all periods

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Abstract

It is proved that if f is a homeomorphism of the two-sphere with an invariant set V of cardinality $N = 4$, then either f has periodic orbits of all periods or it belongs to one of a small number of algebraically finite isotopy classes relative to V. For $N < 4$, the second case always holds. On the other hand, for each $N \geqslant 7$ we give examples of pseudo-Anosov homeomorphisms of the sphere, relative to a set of N points, for which not all periods occur.

1. *Introduction*

Following the classic papers of [**11**] and [**17**] about the set of periods for maps of the interval, it is natural to ask whether there are simple conditions on two-dimensional maps which guarantee that periodic orbits of all periods occur. In this paper we investigate the case of homeomorphisms of surfaces of genus zero.

Some results have already been obtained on this problem. In particular, [**5**] gave an example of a period-3 braid on the disc which implies all periods, and [**10**] generalized this to show that every orientation-preserving homeomorphism of a disc with a period-3 orbit either has all periods or is isotopic (relative to the period-3 orbit) to a rotation. We prove an analogous result for all homeomorphisms of the sphere with an invariant set of cardinality four. This covers both orientation-preserving and orientation-reversing cases, and the set of cardinality four can be either a fixed point and a period-3 orbit, or four fixed points, or two fixed points and a period-2 orbit, or two period-2 orbits, or a period-4 orbit. By blowing up some of these orbits into discs, removing them and checking to see whether there still remain periodic orbits of the given period, we extend our result to most homeomorphisms of surfaces of genus zero with boundary, relative to a finite invariant set, when the sum of the numbers of boundary components and points in the invariant set is four. In particular, we recover Kolev's result.

The key tool is the Nielsen–Thurston classification of surface homeomorphisms up to isotopy (see e.g. [**8**] or [**2**]). Let M be an oriented compact surface, possibly with boundary and possibly with a finite number of distinguished points, called *punctures*.